University of Illinois at Urbana–Champaign
Department of Mathematics

First Midterm Exam for **MATH/CS-313** (Combinatorial Mathematics), Section F1
Instructor: M.N. Kolountzakis

February 24, 1997

YOUR NAME:

SCORE:

1	
2	
3	
4	
5	
TOTAL	

No books, notes or calculators allowed
Maximum number of points: 50
Duration: 1 hour

*** Explain all your answers ****

GOOD LUCK!

1 (10 points) Show that the number of pairs (A_1, A_2), where

$$A_1, A_2 \subseteq [n], \quad \text{and} \quad A_1 \cap A_2 = \emptyset,$$

is 3^n.

2 Let F_n, $n \geq 0$, be the Fibonacci numbers defined by $F_0 = F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. Without using the closed formula for F_n and using only the above definition show:

(a) (6 points) $F_n^2 - F_{n+1}F_{n-1} = (-1)^n$, for $n \geq 1$.

(b) (4 points) $\sum_{i=0}^{n} F_i = F_{n+2} - 1$, for $n \geq 0$.

(HINT: Use induction.)

3 (10 points) You are reminded of the Principle of Inclusion and Exclusion in the following form.

Theorem 1 *Let $F, G : \mathcal{P}([n]) \to \mathbf{R}$ be two functions defined on the subsets of $[n]$. Then the following are equivalent:*
(a) $G(I) = \sum_{J \subseteq I} F(J)$, for all $I \subseteq [n]$.
(b) $F(I) = \sum_{J \subseteq I} (-1)^{|I \setminus J|} G(J)$, for all $I \subseteq [n]$.

Prove that if $f, g : \{0, \ldots, n\} \to \mathbf{R}$ then

$$g(i) = \sum_{j \leq i} \binom{i}{j} f(j), \quad \text{for all } i,$$

implies

$$f(i) = \sum_{j \leq i} (-1)^{i-j} \binom{i}{j} g(j), \quad \text{for all } i.$$

4 (10 points) Let $A_1, \ldots, A_m$ and $B_1, \ldots, B_m$ be partitions of $[mn]$ (i.e., the A_i are disjoint and their union is $[mn]$, and similarly for the B_i), with $|A_i| = |B_i| = n$, for all i.

Show that, after renumbering the B's, we can have

$$A_i \cap B_i \neq \emptyset, \quad \text{for all } i.$$

(HINT: Use Hall's theorem.)

5 (10 points) Let $A \subseteq \{1, 2, \ldots, 2n\}$ with $|A| > n$. Show that there are $a, b \in A$, with $a < b$, such that a divides b.

HINT: Use the Pigeonhole Principle. Partition $[2n]$ into n subsets B_i, $i = 1, \ldots, n$, with the property

$$x, y \in B_i \ \ \& \ \ x < y \implies x \text{ divides } y.$$

SCRATCH PAGE