University of Illinois at Urbana–Champaign
Department of Mathematics

Final Exam for **MATH/CS-313** (Combinatorial Mathematics), Section F1
Instructor: M.N. Kolountzakis

May 14, 1997

YOUR NAME:

SCORE:

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL	

No books, notes or calculators allowed
Maximum number of points: 100
Duration: 3 hours

*** Explain all your answers ****

GOOD LUCK!

1 (10 points) Consider the 8×8 chessboard. In how many ways can you pick a white and a black square that do not share a corner?

2 (10 points) What is the coefficient of $x^{10}y^5z^6$ in the expansion of $(x+y+z)^{21}$ and why?

3 (10 points) Find a formula for the expressions

$$\sum_{i=0}^{k} \binom{n+i}{i} \quad \text{and} \quad \sum_{k=1}^{n} k \binom{n}{k}.$$

4 (10 points) A permutation π of $[n]$ is called *even* if it can be written as the product of an even number of transpositions (otherwise it is called *odd*). Show that there are as many even permutations of $[n]$ as there are odd.

5 (10 points) If the incidence matrix A of the family $A_1, \ldots, A_n \subseteq [n]$ is invertible show that the family has a SDR.

HINT: Use induction on n. Show that you may assume that $A_1 = \{x_1\}$.

6 (10 points) Let n be even and $\mathcal{F}$ be a family of subsets of $[n]$ with the Sperner property, i.e., there is no element of $\mathcal{F}$ that contains another. Show that $|\mathcal{F}| \leq \binom{n}{n/2}$ and that equality is possible.

7 (10 points) A *principal* submatrix of a square matrix is the matrix you get if you keep a subset of the rows and the <u>same</u> subset of the columns. Show that there is a function $f(n)$ such that every $f(n) \times f(n)$ matrix with 0-1 entries has a principal $n \times n$ submatrix with all the elements above the diagonal being equal and all the elements below the diagonal being equal (we do not care what happens *on* the diagonal).

HINT: Ramsey.

8 (10 points) Let M be a connected multigraph (without loops but possibly with multiple edges) whose every vertex has even degree. Show that one can assign a direction to every edge in such a way that in every vertex there are as many edges coming in as there are edges emanating from it.

HINT: Can M be a tree? Use induction on the number of edges.

9 (10 points) Prove that K_5, the complete graph on 5 vertices, is not planar.

HINT: Use Euler's formula $V + F = E + 2$ and argue about how many edges are needed to bound all the faces.

10 (10 points) Let A be a Hadamard matrix of order $n > 2$. Show that n is divisible by 4.

SCRATCH PAGE