University of Illinois at Urbana–Champaign
Department of Mathematics

Final Exam for **MATH-315** (Linear Algebra), Section E2
Instructor: M.N. Kolountzakis

May 16, 1997

YOUR NAME:

SCORE:

1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
TOTAL	

No books, notes or calculators allowed
Maximum number of points: 100
Duration: 3 hours

**** Explain all your answers ****

GOOD LUCK!

1 (10 points) Find all the solutions (if any) of the following linear system (which is given in augmented matrix form)

$$\left(\begin{array}{rrr|r} 1 & -1 & 1 & 0 \\ -1 & 1 & -1 & 0 \\ 4 & 2 & 0 & 8 \\ 0 & 2 & 5 & 9 \end{array}\right)$$

2 (10 points) In P_4 (the vector space of all polynomials of degree up to 3) find a basis for the subspace spanned by the polynomials

$$x,\ x-1,\ x^3+1,\ x^3-1.$$

3 (10 points) Let $\mathbf{b}_1 = (1,0,0)^\top$, $\mathbf{b}_2 = (1,1,0)^\top$, and $\mathbf{b}_3 = (1,1,1)^\top$. Find the transition matrix from the standard basis $\mathcal{E}$ of $\mathbf{R}^3$ to the basis $\mathcal{B} = [\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3]$.

What are the coordinates of $\mathbf{x} = (1,2,3)^\top$ with respect to $\mathcal{B}$?

What are the coordinates of $\mathbf{y} = \mathbf{b}_1 + \mathbf{b}_2 + \mathbf{b}_3$ with respect to $\mathcal{E}$?

4 (10 points) The linear transformation $L : P_3 \to P_2$ is defined by

$$L(p(x)) = p'(x) + p(0)$$

(p' is the derivative of the polynomial p). Find the matrix representation of L with respect to the ordered bases $[x^2, x, 1]$ and $[2, 1-x]$.

What are the coordinates of $L(x^2+1)$ with respect to the basis $[2, 1-x]$?

5 (10 points) Find <u>all</u> least squares solutions for the system $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{pmatrix} 1 & 1 & 3 \\ -1 & 3 & 1 \\ 1 & 2 & 4 \end{pmatrix},$$

and $\mathbf{b} = (-2, 0, 8)^\top$.

6 (10 points) Let S be a subspace of $\mathbf{R}^n$ and $S^\perp$ its orthogonal complement. Let $\mathbf{x} \in \mathbf{R}^n$ and assume that

$$\mathbf{x} \;=\; \mathbf{u}_1 + \mathbf{v}_1 \;=\; \mathbf{u}_2 + \mathbf{v}_2,$$

with $\mathbf{u}_1, \mathbf{u}_2 \in S$ and $\mathbf{v}_1, \mathbf{v}_2 \in S^\perp$.

Show that $\mathbf{u}_1 = \mathbf{u}_2$ and $\mathbf{v}_1 = \mathbf{v}_2$.

7 (10 points) Find an orthonormal basis for the subspace of $\mathbf{R}^4$ spanned by the three vectors $\mathbf{x}_1 = (4, 2, 2, 1)^\top$, $\mathbf{x}_2 = (2, 0, 0, 2)^\top$ and $\mathbf{x}_3 = (1, 1, -1, 1)^\top$.

8 (10 points) Find the eigenvalues and corresponding eigenspaces for the matrix

$$A = \begin{pmatrix} 1 & 2 & 1 \\ 0 & 3 & 1 \\ 0 & 5 & -1 \end{pmatrix}.$$

9 (10 points) Let $A \in \mathbf{R}^{n \times n}$ be diagonalizable and assume that all its eigenvalues are either 1 or -1. Show that $A^{-1} = A$.

10 (10 points) For $\mathbf{x} \in \mathbf{R}^n$ we define the norms

$$\|\mathbf{x}\|_1 = |x_1| + \cdots + |x_n|,$$

and

$$\|\mathbf{x}\|_2 = \sqrt{|x_1|^2 + \cdots + |x_n|^2}.$$

Prove that $\|\mathbf{x}\|_1 \leq \sqrt{n} \cdot \|\mathbf{x}\|_2$.

HINT: Use the Cauchy-Schwartz inequality:

$$(a_1 b_1 + \cdots + a_n b_n)^2 \leq (a_1^2 + \cdots + a_n^2)(b_1^2 + \cdots + b_n^2).$$

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