These problems are to be solved by you as part of your preparation for the midterm exam.

1. Compute the real and imaginary part of $\dfrac{i-4}{2i-3}$.

2. Compute the modulus and the conjugate of the numbers $(1+i)^6,\ i^{17}$.

3. Write the following numbers in the form $x+iy$.

$$1/(1-i),\ 1/(2\sqrt{3}-2i),\ \frac{1+4i}{3+2i},\ e^{-2\pi i/3},\ 3e^{2\pi i/4}.$$

4. Write the following numbers in polar form $re^{i\theta}$.

$$8,\ 6i,\ \left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)^7.$$

5. Let $z = 1+2i,\ w = 2-i$. Compute

$$z+3w,\ \overline{w}-z,\ z^3,\ \mathrm{Re}\,(w^2+w),\ z^2+\overline{z}+i.$$

6. Compute the square roots of $-1-i$.

7. Compute the cube roots of -8.

8. Prove that there is no $z \in \mathbb{C}$ such that $|z| - z = i$.

9. Find all $z \in \mathbb{C}$ such that $z^2 \in \mathbb{R}$.

10. Simplify the following numbers

$$i^{999},\ i^3+i^6+i^9,\ \frac{1+2i}{3-4i} - \frac{2-i}{5i^3},\ \frac{|1+i|}{1-i}.$$

11. Solve the following equations

$$z^2+5=0,\ z^2-4z+5=0,\ z^4+1=0,\ 32z^5-1=0,\ z+i=\frac{1}{z}+\frac{1}{i}.$$

12. Find an expression for $\sin(3\theta)$ in terms of $\sin\theta, \cos\theta$.

13. Prove the parallelogram law

$$|z-w|^2 + |z+w|^2 = 2(|z|^2+|w|^2).$$

14. Describe the regions of $\mathbb{C}$ described by each of the following relations

$$|z-(3+2i)| = 5,\ |z-2+i| = |1+3i|,\ |z+2i| = 2,\ |z-4| = 0.$$

15. Describe the regions of $\mathbb{C}$ described by each of the following relations

$$|z+3| = |z-4i|,\ |z+1| = 2|z-1|,\ \frac{\pi}{4} \le \mathrm{Arg} z \le \frac{\pi}{2},\ \mathrm{Arg}\frac{z-1}{z+1} = \frac{\pi}{4}.$$

16. Describe the regions of $\mathbb{C}$ described by each of the following relations

$$|z+3| < 2,\ |\mathrm{Im}\,z| < 1,\ 1 < |z-1| < 2,\ |z-1|+|z+1| \le 2.$$

17. Which of the following functions of $z = x+iy$ satisfy the Cauchy-Riemann equations?

$$e^{-x}e^{-iy},\ 2x+ixy^2,\ x^2+iy^2,\ e^x e^{-iy},\ \mathrm{Im}\,z, |z|^2,\ \overline{z}.$$

18. We define

$$\cosh z = (e^z+e^{-z})/2,\ \sinh z = (e^z-e^{-z})/2.$$

Prove

$$\begin{aligned}
\sin z &= \sin x\cosh y + i\cos x\sinh y\\
\cos z &= \cos x\cosh y - i\sin x\sinh y\\
|\sin z|^2 &= \sin^2 x + \sinh^2 y\\
|\cos z|^2 &= \cos^2 x + \sinh^2 y
\end{aligned}$$

19. Find the radius of convergence of each of the following power series.

$$\sum_{k\ge 0} \cos k z^k,\ \sum_{k\ge 0} 4^k (z-2)^k,\ \sum_{k\ge 0} \frac{z^k}{k^k},\ \sum_{k\ge 1} \frac{(-1)^k}{k} z^{k(k+1)},\ \sum_{k\ge 0} z^{k!}.$$

20. Find a simple formula for each of the following power series.

$$\sum_{k\ge 0} \frac{z^{2k}}{k!},\ \sum_{k\ge 1} k(z-1)^{k-1},\ \sum_{k\ge 2} k(k-1)z^k.$$

21. For which z does the series $\sum_{n=0}^{\infty} z^n$ converge?

22. Assuming as known the Taylor series $e^z = \sum_{n=0}^{\infty} z^n/n!$ find the Taylor series around 0 for $\sin z$, $\cos z$.

23. Assuming as known the Taylor series $e^z = \sum_{n=0}^{\infty} z^n/n!$ show that e^z is the derivative of itself. Similarly compute the derivatives of $\sin z, \cos z$ from their Taylor series.

24. Compute the integral $\oint_{ABC} \overline{z}\, dz$, where ABC is the triangle that connects the points $A=0, B=1, C=i$ in the positive orientation. Similarly compute $\oint_{ABC} \operatorname{Re} z\, dz$. Do the above integrals depend or not on which is the closed curve of integration?

25. Compute the following integrals where C is the square with vertices at $\pm 4 \pm 4i$, in the positive orientation.

$$\oint_C \frac{e^z}{z^3} dz,\ \oint_C \frac{e^z}{(z-\pi i)^2} dz,\ \oint_C \frac{\sin 2z}{(z-\pi)^2} dz,\ \oint_C \frac{e^z \cos z}{(z-\pi)^3} dz.$$

How do the answers to the above change if curve C is the same square but with 2 turns?

26. Compute the integral $\oint_C \frac{z}{z^2+4} dz$ along the circle $|z|=3$ in the positive orientation.

27. If f is analytic in the closed disk $|z-a| \le r$ show that $f(a) = \frac{1}{2\pi} \int_0^{2\pi} f(a+re^{i\theta})\, d\theta$ (mean value property).

28. After you find the constants A and B such that

$$\frac{1}{z^2+1} = \frac{A}{z+i} + \frac{B}{z-i}$$

compute $\oint_{|z|=2} \frac{ze^z}{z^2+1} dz$.

29. We know Cauchy's integral formula $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(w)}{w-z} dw$ if f is analytic in the interior of C and on it. Show that the same holds if f is analytic in the interior of C and only continuous in the closed set that consists of C and its interior. For simplicity you may assume that C is a circle and z a point in its interior.

30. A trigonometric polynomial is a function $f : [0, 2\pi) \to \mathbb{C}$ of the form

$$f(t) = \sum_{n=-N}^{N} a_n e^{int}, \quad t \in [0, 2\pi).$$

The natural number N is called the degree of the polynomial (we assume $a_N \ne 0$ or $a_{-N} \ne 0$). At most how many roots can a trigonometric polynomial of degree N have in the interval $[0, 2\pi)$?