Ασκήσεις για να προετοιμαστείτε για το διαγώνισμα

3/5/12

Από βιβλίο K. Stromberg, Probability for Analysts

Από Κεφ. 3

5. Prove that if the family $\{A, B, C\}$ of three distinct events is independent, then so is $\{A, B \cup C\}$.

7. Suppose $Y : \Omega \mapsto \mathbb{R}^d$ is a random variable.

$$\mu := \operatorname{dist} Y, \quad \text{and} \quad \mu(B) = 0 \text{ or } 1 \qquad \forall B \in \mathscr{B}^d.$$

Prove that $\mu = \delta_c$ for some $c \in \mathbb{R}^d$. In this case we say that Y is **almost surely a constant**.

[*Hint*: Take a countable base $\mathfrak{U}$ for the topology of $\mathbb{R}^d$. Let

$$\mathfrak{U}_0 := \{V \in \mathfrak{U} : \mu(V) = 0\}.$$

Show that the union of $\mathfrak{U}_0$ contains all but one point of $\mathbb{R}^d$.]

8. Suppose $X : \Omega \mapsto \mathbb{R}^n$ is a random variable, $f : \mathbb{R}^n \mapsto \mathbb{R}^d$ is Borel measurable, and $Y := f \circ X$. Prove that $\{X, Y\}$ is independent if and only if Y is almost surely a constant.

13. Give an example of two real random variables X and Y on the same probability space such that $\{X, Y\}$ is *not* independent, but yet $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

14. Let Z and W be integrable complex random variables: Z, $W \in \mathbf{L}_1(\Omega, \mathcal{A}, P)$. Prove that

(a) $\mathbb{V}(Z + c) = \mathbb{V}(Z) \quad \forall c \in \mathbb{C}$,

(b) $\mathbb{V}(Z + W) + \mathbb{V}(Z - W) = 2\mathbb{V}(Z) + 2\mathbb{V}(W)$, and

(c) if $\{Z, W\}$ is independent, then $\mathbb{V}(Z+W) = \mathbb{V}(Z)+\mathbb{V}(W)$ even if some of these variances are $+\infty$.

[*Hint*: For (b) and (c), first suppose $\mathbb{E}(Z) = 0 = \mathbb{E}(W)$. Then use (a) for the general case.]

17. Consider the random quadratic equation $Ax^2 + Bx + C = 0$, where $\{A, B, C\}$ is an independent set of real-valued random variables all having the same distribution μ and $\mu(\{0\}) = 0$.

(a) Prove that the probability p that the roots of this equation are real is given by

$$p = \int_{\mathbb{R}} \int_{\mathbb{R}} \mu(]-\infty, y^2/4x]) \, d\mu(y) \, d\mu(x).$$

[*Hint*: Use Exercise 16.]

(b) Calculate p if μ is Lebesgue measure on $[0, 1]$.

(c) Calculate p if μ is normalized Lebesgue measure on $[-1, 1]$:

$$\mu(B) = \frac{1}{2}\lambda(B \cap [-1, 1]) \qquad \forall B \in \mathscr{B}^1.$$

20. Suppose $\{N, X_1, X_2, \dots\}$ is an independent set of random variables such that $N\colon \Omega \mapsto \mathbb{N}$, $X_j : \Omega \mapsto \mathbb{R}^d$ with

$$\operatorname{dist} X_j = \mu_j, \quad \text{and} \quad P(N = n) = p_n \qquad \forall j, n \in \mathbb{N}.$$

Define $S\colon \Omega \mapsto \mathbb{R}^d$ by $S(\omega) := \sum_{j=1}^{N(\omega)} X_j(\omega)$ and write

$$\pi_n := \mathop{\ast}_{j=1}^{n} \mu_j \qquad (n \in \mathbb{N}).$$

Prove that S is a random variable and the distribution σ of S is given by $\sigma(B) = \sum_{n=1}^{\infty} p_n \pi_n(B) \quad \forall B \in \mathscr{B}^d$.

Από Κεφ. 4

2. Let $\{U_n\}_{n=1}^{\infty}$ be an independent sequence of real random variables with $U_n \geqq 0 \quad \forall n$. Define $V_n := 1 \wedge U_n$, the pointwise minimum of 1 and U_n. Use (4.13) to prove that $\sum_{n=1}^{\infty} U_n < \infty$ a.s. $\iff \sum_{n=1}^{\infty} \mathbb{E}(V_n) < \infty$.

3. Let $\{U_n\}_{n=1}^{\infty}$ be as in Exercise 2.

(a) Prove that $\sum_{n=1}^{\infty} \mathbb{E}(U_n) < \infty \implies \sum_{n=1}^{\infty} U_n < \infty$ a.s.

(b) Give an example to show that the converse of the implication in (a) may fail.

6. Let $\{X_n\}$ be an independent sequence of symmetric real random variables that all have the same distribution μ and let $(a_n)_{n=1}^{\infty}$ be a sequence

of positive real numbers. Prove that $\sum_{n=1}^{\infty} a_n X_n$ converges a.s. if and only if both

$$\sum_{n=1}^{\infty} a_n^2 \int_{[0,a_n^{-1}]} x^2 \, d\mu(x) < \infty \quad \textit{and} \quad \sum_{n=1}^{\infty} \mu(]a_n^{-1}, \infty[) < \infty.$$

8. Let $\{A_n\}_{n=1}^{\infty}$ be an independent sequence of events with

$$\lim_{n\to\infty} P(A_n) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} P(A_n) = \infty.$$

Prove that there is *no* sequence $(c_n)_{n=1}^{\infty} \subset \mathbb{R}$ for which

$$P\left(\sum_{n=1}^{\infty} (c_n - \mathbf{1}_{A_n}) \text{ converges}\right) > 0.$$

11. Let $\{X_n\}_{n=1}^{\infty}$ be an independent sequence of real random variables and put $X := \bigvee_{n=1}^{\infty} X_n$. Prove that

(a) $P(X < \infty)$ is 0 or 1,

(b) $P(X < \infty) = 1 \iff \sum_{n=1}^{\infty} P(X_n > c) < \infty$ for some $c \in \mathbb{R}$,

(c) $P(X \leqq t) = \prod_{n=1}^{\infty} P(X_n \leqq t) \quad \forall t \in \mathbb{R}$.