5. Prove that if the family \( \{A, B, C\} \) of three distinct events is independent, then so is \( \{A, B \cup C\} \).

7. Suppose \( Y: \Omega \mapsto \mathbb{R}^d \) is a random variable.

\[
\mu := \text{dist } Y, \quad \text{and} \quad \mu(B) = 0 \text{ or } 1 \quad \forall B \in \mathcal{B}^d.
\]

Prove that \( \mu = \delta_c \) for some \( c \in \mathbb{R}^d \). In this case we say that \( Y \) is almost surely a constant.

[Hint: Take a countable base \( \mathcal{U} \) for the topology of \( \mathbb{R}^d \). Let

\[
\mathcal{U}_0 := \{V \in \mathcal{U} : \mu(V) = 0\}.
\]

Show that the union of \( \mathcal{U}_0 \) contains all but one point of \( \mathbb{R}^d \).]

8. Suppose \( X: \Omega \mapsto \mathbb{R}^n \) is a random variable, \( f: \mathbb{R}^n \mapsto \mathbb{R}^d \) is Borel measurable, and \( Y := f \circ X \). Prove that \( \{X, Y\} \) is independent if and only if \( Y \) is almost surely a constant.
13. Give an example of two real random variables $X$ and $Y$ on the same probability space such that \{X, Y\} is \textit{not} independent, but yet $\mathbb{E}(XY) = \mathbb{E}(X)\mathbb{E}(Y)$.

14. Let $Z$ and $W$ be integrable complex random variables: $Z, W \in L_1(\Omega, \mathcal{A}, P)$. Prove that
   
   (a) $\mathbb{V}(Z + c) = \mathbb{V}(Z) \quad \forall c \in \mathbb{C},$
   
   (b) $\mathbb{V}(Z + W) + \mathbb{V}(Z - W) = 2\mathbb{V}(Z) + 2\mathbb{V}(W)$, and
   
   (c) if \{Z, W\} is independent, then $\mathbb{V}(Z + W) = \mathbb{V}(Z) + \mathbb{V}(W)$ even if some of these variances are $+\infty$.

   \textit{[Hint: For (b) and (c), first suppose $\mathbb{E}(Z) = 0 = \mathbb{E}(W)$. Then use (a) for the general case.]}  

17. Consider the random quadratic equation $Ax^2 + Bx + C = 0$, where \{A, B, C\} is an independent set of real-valued random variables all having the same distribution $\mu$ and $\mu(\{0\}) = 0$.

   (a) Prove that the probability $p$ that the roots of this equation are real is given by
   
   $$p = \int_{\mathbb{R}} \int_{\mathbb{R}} \mu(\left[ -\infty, y^2/4x \right] ) \, d\mu(y) \, d\mu(x).$$

   \textit{[Hint: Use Exercise 16.]}  

   (b) Calculate $p$ if $\mu$ is Lebesgue measure on $[0, 1]$.

   (c) Calculate $p$ if $\mu$ is normalized Lebesgue measure on $[-1, 1]$:

   $$\mu(B) = \frac{1}{2} \lambda(B \cap [-1, 1]) \quad \forall B \in \mathcal{B}^1.$$
20. Suppose \( \{N, X_1, X_2, \ldots \} \) is an independent set of random variables such that \( N: \Omega \mapsto \mathbb{N}, X_j: \Omega \mapsto \mathbb{R}^d \) with
\[
\text{dist } X_j = \mu_j, \quad \text{and} \quad P(N = n) = p_n \quad \forall j, n \in \mathbb{N}.
\]
Define \( S: \Omega \mapsto \mathbb{R}^d \) by \( S(\omega) := \sum_{j=1}^{N(\omega)} X_j(\omega) \) and write
\[
\pi_n := \prod_{j=1}^n \mu_j \quad (n \in \mathbb{N}).
\]
Prove that \( S \) is a random variable and the distribution \( \sigma \) of \( S \) is given by
\[
\sigma(B) = \sum_{n=1}^{\infty} p_n \pi_n(B) \quad \forall B \in \mathcal{B}^d.
\]

Από Κεφ. 4

2. Let \( \{U_n\}_{n=1}^{\infty} \) be an independent sequence of real random variables with \( U_n \geq 0 \ \forall n \). Define \( V_n := 1 \wedge U_n \), the pointwise minimum of 1 and \( U_n \). Use (4.13) to prove that \( \sum_{n=1}^{\infty} U_n < \infty \) a.s. \( \iff \sum_{n=1}^{\infty} \mathbb{E}(V_n) < \infty \).

3. Let \( \{U_n\}_{n=1}^{\infty} \) be as in Exercise 2.

(a) Prove that \( \sum_{n=1}^{\infty} \mathbb{E}(U_n) < \infty \implies \sum_{n=1}^{\infty} U_n < \infty \) a.s.

(b) Give an example to show that the converse of the implication in (a) may fail.
6. Let \( \{X_n\} \) be an independent sequence of symmetric real random variables that all have the same distribution \( \mu \) and let \( (a_n)_{n=1}^{\infty} \) be a sequence of positive real numbers. Prove that \( \sum_{n=1}^{\infty} a_n X_n \) converges a.s. if and only if both

\[
\sum_{n=1}^{\infty} a_n^2 \int_{[0,a_n^{-1}]} x^2 \, d\mu(x) < \infty \quad \text{and} \quad \sum_{n=1}^{\infty} \mu([a_n^{-1}, \infty]) < \infty.
\]

8. Let \( \{A_n\}_{n=1}^{\infty} \) be an independent sequence of events with

\[
\lim_{n \to \infty} P(A_n) = 0 \quad \text{and} \quad \sum_{n=1}^{\infty} P(A_n) = \infty.
\]

Prove that there is no sequence \( (c_n)_{n=1}^{\infty} \subset \mathbb{R} \) for which

\[
P\left( \sum_{n=1}^{\infty} (c_n - 1_{A_n}) \text{ converges} \right) > 0.
\]

11. Let \( \{X_n\}_{n=1}^{\infty} \) be an independent sequence of real random variables and put \( X := \bigvee_{n=1}^{\infty} X_n \). Prove that

(a) \( P(X < \infty) = 0 \) or 1,
(b) \( P(X < \infty) = 1 \iff \sum_{n=1}^{\infty} P(X_n > c) < \infty \) for some \( c \in \mathbb{R} \),
(c) \( P(X \leq t) = \prod_{n=1}^{\infty} P(X_n \leq t) \quad \forall t \in \mathbb{R} \).