Solutions of some exercises



**1.** If a sequence of polynomials convergers to $f$ uniformly on $\mathbb{R}$ show that $f$ too is a polynomial.

**Solution**:

Suppose the polynomials $p_n(x) \to f(x)$ uniformly on $\mathbb{R}$. Suppose $n_0$ is such that $|p_n(x) - f(x)| \le 1$ for all $x$. Then for $m, n \ge n_0$ we have, by the triangle inequality, that $|p_m(x) - p_n(x)| \le 2$. But the difference of two polynomials is a polynomial and the only polynomials which are bounded on $\mathbb{R}$ are the constants, so $p_m(x) - p_n(x) = C_{m,n}$. In other words the non-constant part of the polynomials $p_n$, $n \ge n_0$, is the same. We can move this part to $f$ to conclude that the sequence of constant terms of our polynomials converges uniformly to a function, hence our constant terms converge to some $C$. It follows that $f$ minus the polynomial we subtracted from it is a constant, hence $f$ is a polynomial.

**2.** If $f : [1, +\infty) \to \mathbb{R}$ is continuous and the limit $\lim_{x\to+\infty} f(x)$ exists and is a real number show that $f$ can be approximated uniformly by functions of the form $p(1/x)$ where $p$ is a polynomial.

**Solution**: The function $g : (0, 1] \to \mathbb{R}$ defined by $g(x) = f(1/x)$ is obviously continuous on $(0, 1]$. If we define it at 0 to be $g(0) = \lim_{x\to\infty} f(x)$ then it becomes continuous on $[0, 1]$. By Werierstrass' theorem $g(x)$ is a uniform limit of polynomials $p_n(x)$, which implies that $f(x)$ is a uniform limit of the functions $p_n(1/x)$.