1. Let X be the space $C^1([a,b])$ of functions with a continuous derivative in $[a,b]$ (side derivatives at the end-points). We define the norm.

$$\|f\| = |f(a)| + \|f'\|_\infty.$$

Show that this is indeed a norm and that X, with this norm, is a complete space.

💡For $x \in [a,b]$ we have $f(x) = f(a) + \int_a^x f'(t)\,dt$.

Solution: The only norm property that is nontrivial to verify is the property

$$\|f\| = 0 \implies f = 0.$$

But $\|f\| = 0$ implies $f(a) = 0$ and $f' = 0$ everywhere. Since $f(x) = f(a) + \int_a^x f'(t)\,dt$ we also obtain that $f = 0$ everywhere.

To prove completeness assume f_n is a Cauchy sequence in our space. Then $f_n(a)$ is a Cauchy sequence in $\mathbb{C}$ and f_n' is a Cauchy sequence in $C([a,b])$. By the completeness of $\mathbb{C}$ and $C([a,b])$ we obtain first that $f_n(a)$ converges to a value $A \in \mathbb{C}$ and that f_n' converges to a function $B(x)$ in $C([a,b])$. We claim that this implies that f_n converges to the function

$$f(x) = A + \int_a^x B(t)\,dt,$$

which is clearly a function in $C^1([a,b])$. Indeed we have

$$\|f_n - f\| = |f_n(a) - f(a)| + \|f_n' - f'\|_\infty,$$

and, since $f(a) = A$ and $f'(t) = B(t)$, we have

$$\|f_n - f\| = |f_n(a) - A| + \|f_n' - B\|_\infty.$$

But both terms above converge to 0.

2. Consider the sequence space $\ell^1(\mathbb{N})$ which consists of all complex sequences $x = (x_1, x_2, \ldots)$ such that

$$\sum_{n=1}^\infty |x_n| < \infty.$$

The norm is $\|x\|_1 = \sum_{n=1}^\infty |x_j|$. Consider the operator $T : \ell^1(\mathbb{N}) \to \ell^1(\mathbb{N})$ defined by

$$Tx = (x_2, x_3, \ldots).$$

Show that it is a bounded operator and find its norm.

Solution: We have

$$\|Tx\|_1 = \sum_{n=2}^\infty |x_n| \le \|x\|_1,$$

so $\|T\| \le 1$.

Choosing $x = (0, 1, 0, \ldots)$, for example, shows that $\|Tx\|_1 = \|x\|_1$ is possible, so that $\|T\| = 1$.

3. Consider the Banach space $\ell^2(\mathbb{N})$ which consists of all complex sequences $x = (x_1, x_2, \ldots)$ such that

$$\sum_{n=1}^\infty |x_n|^2 < \infty.$$

The norm is $\|x\| = \left(\sum_{n=1}^\infty |x_n|^2\right)^{1/2}$.

If the series $\sum_{n=1}^\infty a_n x_n$ converges for every $x \in \ell^2(\mathbb{N})$ (i) show that

$$Tx = \sum_{n=1}^\infty a_n x_n$$

is a bounded linear functional $\ell^2(\mathbb{N}) \to \mathbb{C}$. (ii) Show also that $\sum_{n=1}^\infty |a_n|^2 < \infty$ (in other words the sequence $a = (a_1, a_2, \ldots)$ is in $\ell^2(\mathbb{N})$).

💡For the first question apply the Banach-Steinhaus theorem to the sequence of functionals

$$T_N x = \sum_{n=1}^N a_n x_n.$$

Solution: (i) Define the linear functionals $\ell^2(\mathbb{N}) \to \mathbb{C}$

$$T_N(x) = \sum_{n=1}^{N} a_n x_n.$$

By the Cauchy-Schwarz inequality they are bounded functionals, with $\|T_N\| \le \left(\sum_{n=1}^{N} |a_n|^2\right)^{1/2}$. For fixed $x \in \ell^2(\mathbb{N})$ we are assuming that $T_N(x)$ converges, hence it is a bounded sequence. By the Banach-Steinhaus theorem we have that, for some finite positive number M,

$$|T_N x| \le M\|x\|.$$

Since $Tx = \lim_N T_N x$ it follows that we also have $|Tx| \le M\|x\|$ and T is a bounded linear functional (the linearity is obvious).

(ii) For $N = 1, 2, \ldots$ define $x_N = (\overline{a_1}, \overline{a_2}, \ldots, \overline{a_N}, 0, 0, \ldots)$. By (i) we have

$$|Tx| \le M\|x\|.$$

But $Tx_n = \sum_{n=1}^{N} |a_n|^2$ and $\|x\| = \left(\sum_{n=1}^{N} |a_n|^2\right)^{1/2}$ so that we have

$$\sum_{n=1}^{N} |a_n|^2 \le M \left(\sum_{n=1}^{N} |a_n|^2\right)^{1/2}$$

or

$$\left(\sum_{n=1}^{N} |a_n|^2\right)^{1/2} \le M.$$

This implies that $\sum_{n=1}^{\infty} |a_n|^2 \le M^2$ as we had to show.