Solutions

1. If $E \subseteq [0, 2\pi]$ and $\xi_n \in \mathbb{R}$ is any sequence show that
$$\int_E \cos^2(nx + \xi_n)\, dx \to \frac{1}{2}|E|.$$

💡Use the Riemann-Lebesgue Lemma .

Solution: Observe first that if ϕ_n is any real sequence and $f \in L^1(\mathbb{T})$ then
$$⨍ f(x)\cos(nx + \phi_n)\, dx \to 0, \qquad \text{as } |n| \to \infty.$$
This is immediate from the Riemann-Lebesgue lemma as the above integral is equal to
$$\frac{1}{2}\left(\widehat{f}(-n)e^{i\phi_n} + \widehat{f}(n)e^{-i\phi_n}\right).$$
Next write
$$\begin{aligned}\cos^2(nx + \xi_n) &= \frac{1}{4}\left(e^{i(nx+\xi_n)} + e^{-i(nx+\xi_n)}\right)\\ &= \frac{1}{2} + \frac{1}{2}\cos(2(nx + \xi_n)),\end{aligned}$$
which implies
$$\int_E \cos^2(nx + \xi_n)\, dx = \int_E \frac{1}{2} + \frac{1}{2}\int \chi_E(x)\cos(2(nx + \xi_n)) = \frac{1}{2}|E| + o(1),$$
as we had to show.

2. If $0 < \alpha < \beta < 1$ construct a function f which is Lip-α but not Lip-β.

Solution: For $\eta \in (0, 1)$ let
$$f_\eta(x) = x^\eta, \qquad \text{for } x \in [0, 1].$$
Then, for $x > 0$, $f_\eta'(x) = \eta x^{\eta-1}$ and by the Mean Value Theorem and for $0 < h < 1$ we have for some $\xi \in [x, x + h]$
$$f_\eta(x + h) - f_\eta(x) = f_\eta'(\xi)h = \eta\xi^{\eta-1}h = \eta\left(\frac{h}{\xi}\right)^{1-\eta} h^\eta.$$
If $h \le x$ then $\left(\frac{h}{\xi}\right)^{1-\eta} \le 1$ so that in that case we have $f_\eta(x + h) - f_\eta(x) \le \eta h^\eta$. If $x < h$ then $f_\eta(x + h) - f_\eta(x) \le (x + h)^\eta \le 2^\eta h^\eta$. So whenever $x > 0$ we have $f_\eta(x + h) - f_\eta(x) \le 2^\eta h^\eta$.

For $x = 0$ we have
$$f_\eta(0 + h) - f_\eta(0) = h^\eta,$$
so for all $x \in [0, 1]$ we have proved that $f_\eta(x + h) - f_\eta(x) \le 2^\eta h^\eta$, so that $f_\eta \in$ Lip-η. Checking the difference $f_\eta(x+h) - f_\eta(x)$ for $x = 0$ shows immediately that this function cannot be in any higher Lip-η' (i.e. for $\eta' > \eta$).

3. If a function $f \in C(\mathbb{T})$ is Lipschitz-α for some $\alpha > 1$ show that the function is necessarily constant.

💡If $x \ne y$ show that $g(x) = g(y)$ writing
$$|g(x) - g(y)| \le |g(x) - g(x + \delta)| + |g(x + \delta) - g(x + 2\delta)| + \cdots + |g(x + (n - 1)\delta) - g(y)|,$$
where $\delta = (y - x)/n$.

Solution: Following the hint, assuming g satisfies the inequality $|g(x + h) - g(x)| \le Mh^\alpha$,
$$|g(x) - g(y)| \le Mn\delta^\alpha = M\frac{|y - x|}{\delta}\delta^\alpha = M|y - x|\delta^{1-\alpha}.$$
The upper bound can be made arbitrarily small be choosing n large so we have proved $g(x) = g(y)$ for arbitrary x, y.

Alternatively one can show that g is differentiable and that the derivative is 0 everywhere, just by applying the definition of the derivative.