1. If $f \in L^1(\mathbb{R})$ show that the Fourier Transform $\widehat{f}$ is uniformly continuous on $\mathbb{R}$.

Solution: It is a standard fact that any *continuous* function $g(x)$ on $\mathbb{R}$ such that the limits $\lim_{x\to\pm\infty} g(x)$ exist is also uniformly continuous. Since, by the Riemann-Lebesgue lemma, this happens for $\widehat{f}$ it is uniformly continuous.

2. If $f \in L^2(\mathbb{R})$ is the Riemann-Lebesgue Lemma valid?

Solution: No. We have seen (Parseval) that the Fourier Transform is an isometry from $L^2(\mathbb{R})$ to $L^2(\mathbb{R})$ that is also onto. In other words, any L^2 function is the Fourier Transform of *some* L^2 function. Since there are L^2 functions (in fact, L^p functions, for any $p \in [1, +\infty]$) which do not tend to 0 at infinity (example: $f(x) = \sum_{n=1}^{\infty} n\chi_{[n,n+\frac{1}{n^{10}}]}(x)$) the Riemann-Lebesgue lemma does not hold for L^2 functions.

3. Show that there exists a not-identically-zero C^∞ function which vanishes outside a bounded interval.
💡Use the function

$$\phi(x) = \begin{cases} e^{-1/x} & 0 < x \\ 0 & x \le 0 \end{cases}$$

Solution: Using the hint, we first show that $\phi(x)$ is smooth. All we have to check is that all its right derivatives at 0 are 0.

It is very easy to prove by induction on n that

$$f^{(n)}(x) = p_n(1/x) f(x),$$

where $p_n(\cdot)$ is a polynomial. Taking the limit as $x \to 0+$ we obtain 0 (as the exponential defeats any polynomial).

To finish the problem consider the function

$$\psi(x) = \phi(x)\phi(1-x),$$

which is clearly C^∞, not identically zero, and vanishes outside $[0, 1]$.

4. Assume $0 \le \theta \le 1$. If $1 \le p_1 \le p_2 \le \infty$ and $\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}$ show that for every $f : \mathbb{R} \to \mathbb{C}$

$$\|f\|_p \le \|f\|_{p_1}^{\theta} \|f\|_{p_2}^{1-\theta}.$$

💡Use Hölder's inequality as follows

$$\|f\|_p = \left\| |f|^\theta \cdot |f|^{1-\theta} \right\|_p \le \cdots$$

Solution: Assume first $p_2 < \infty$. Then we also have $p_1 \le p < \infty$ and

$$\begin{aligned} \|f\|_p^p &= \int |f|^{\theta p} |f|^{(1-\theta)p} \\ &\le \left\| |f|^{\theta p} \right\|_{\frac{p_1}{\theta p}} \cdot \left\| |f|^{(1-\theta)p} \right\|_{\frac{p_2}{(1-\theta)p}} \qquad \text{(Hölder, with the conjugate exponents } \frac{p_1}{\theta p}, \frac{p_2}{(1-\theta)p}) \\ &= \left(\int |f|^{p_1} \right)^{\frac{\theta p}{p_1}} \left(\int |f|^{p_2} \right)^{\frac{(1-\theta)p}{p_2}}. \end{aligned}$$

Raising to the powet $1/p$ we get the desired inequality.

If $p_2 = +\infty$ and $p_1 = \theta p < \infty$ (otherwise the inequality to be proved is an obvious equality) we have

$$\|f\|_p^p = \int |f|^{\theta p} |f|^{(1-\theta)p} \le \int |f|^{\theta p} \, \|f\|_\infty^{(1-\theta)p}$$

and raising to the power $1/p$ we get

$$\|f\|_p \le \|f\|_{p_1}^{\theta} \|f\|_\infty^{1-\theta}.$$