Proc. Amer. Math. Soc. 120 (194), 1, 157-163.

On Nonnegative Cosine Polynomials With Nonnegative, Integral Coefficients

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Dedicated to S. K. Pichorides

April 1992

Abstract

We show that there is $p_0 > 0$ and $p_1, \ldots, p_N$ non-negative integers, such that

$$0 \le p(x) = p_0 + p_1 \cos x + \cdots + p_N \cos Nx$$

and

$$p_0 \ll s^{1/3}$$

for $s = \sum_{j=0}^{N} p_j$, improving on a result of Odlyzko who showed the existence of such a polynomial p that satisfies $p_0 \ll (s \log s)^{1/3}$. Our result implies an improvement of the best known estimate for a problem of Erdös and Szekeres. As our method is non-constructive, we also give a method for constructing an infinite family of such polynomials, given *one* good "seed" polynomial.

1 Introduction

We consider non-negative cosine polynomials of the form

$$0 \le p(x) = p_0 + p_1 \cos x + p_2 \cos 2x + \cdots + p_N \cos Nx\ , \quad x \in [0, 2\pi]$$

where $p_j \ge 0$. We also write $\widehat{p}(0) = p_0$. Notice that $p(0) = \sum_{j=0}^{N} p_j$ is the maximum of $p(x)$. We are interested in estimating the size of

$$M(s) = \inf_{p(0) \ge s} \widehat{p}(0)$$

for $s \to \infty$. That is, we want to find polynomials of the above form for which $p_0 = \frac{1}{2\pi}\int_0^{2\pi} p(x)dx$ is small compared to the maximum of $p(x)$. In what follows C denotes an arbitrary positive constant and $a \ll b$ means $a \le Cb$ for some C.

If no more restrictions are imposed on the cosine polynomial $p(x)$ then $M(s) = 0$ for all s. This is because the Fejér kernel

$$K_A(x) = \sum_{j=-A}^{A} \left(1 - \frac{|j|}{A+1}\right) e^{ijx} = 1 + \sum_{j=1}^{A} 2\left(1 - \frac{j}{A+1}\right) \cos jx$$

has constant coefficient 1, has $K_A(0) \gg A$ and is non-negative.

If we restrict the coefficients $p_1, \ldots, p_N$ to be either 0 or 1 we have the classical *cosine problem*, about which we know that for some $\epsilon > 0$

$$2^{\log^\epsilon s} \ll M(s) \ll s^{1/2} \tag{1}$$

The upper bound in (1) is easily proved by considering the polynomial

$$f(x) = (\sum_1^A \cos 3^j x)^2 \tag{2}$$

$$= A + \frac{1}{2}\sum_{j=1}^{A} \cos(2 \cdot 3^j x) + \sum_{\substack{k,l=1 \\ k>l}}^{A} \left(\cos(3^k + 3^l)x + \cos(3^k - 3^l)x\right) \tag{3}$$

All cosines in (3) have distinct frequencies. Define $f_1(x) = f(x) + \frac{1}{2}A - \frac{1}{2}\sum_{j=1}^{A}\cos(2\cdot 3^j x)$. Then $f_1(x) \ge 0$, $f_1(0) \gg A^2$, $\widehat{f_1}(0) \ll A$ and f_1 has non-constant coefficients which are either 0 or 1. The lower bound in (1) is much harder to prove and is due to Bourgain [3]. Earlier, Roth [8] had obtained $M(s) \gg (\log s / \log\log s)^{1/2}$.

From this point on, we will study the case of $p_1, \ldots, p_N$ being arbitrary non-negative integers. This case was studied by Odlyzko [7] who showed that

$$M(s) \ll (s \log s)^{1/3} \tag{4}$$

The method is the following. Consider the non-negative polynomial

$$q(x) = \alpha K_A(x) = q_0 + q_1 \cos x + \cdots + q_A \cos Ax$$

whose coefficients are not necessarily integers ($\alpha > 0$). We modify q so that its non-constant coefficients are integers, by adding to it a random polynomial

$$r(x) = r_1 \cos x + \cdots + r_A \cos Ax$$

The coefficients r_j are independent random variables which take values such that $r_j + q_j$ is always an integer. A theorem of Salem and Zygmund [9] guarantees that $\|r\|_\infty$ is small with high probability, and the non-negative polynomial $p(x) = q(x) + r(x) + \|r\|_\infty$ achieves (4) when α is appropriately chosen as a function of A.

Odlyzko studied this problem in connection with a problem posed by Erdös and Szekeres [4]. The problem is to estimate

$$E(n) = \inf \max_{|z|=1} \left| \prod_{k=1}^{n} (1 - z^{a_k}) \right|$$

where $a_1, \ldots, a_n$ may be any positive integers. The following inequality holds (see [7])

$$\log E(n) \ll M(n) \log(n) \tag{5}$$

so that Odlyzko's result implies $\log E(n) \ll n^{1/3} \log^{4/3} n$.

In this paper we replace the random modification in Odlyzko's argument with a more careful modification, based, again, partly on randomization. We use a recent theorem of Spencer [10] which in some cases does better than the Salem-Zygmund theorem. We show in Section (3) that, when $p_1, \ldots, p_N$ are restricted to be non-negative integers, we have

$$M(s) \ll s^{1/3}$$

By (5) this implies $\log E(n) \ll n^{1/3} \log n$. Our method is similar to that used by Beck [2] on a different problem, posed by Littlewood. [1]

Both the Salem-Zygmund theorem and Spencer's theorem are non-constructive. In Section 4 we give a deterministic procedure which, given a polynomial $p(x)$ with non-negative integral Fourier coefficients (in other words p_j is a non-negative even integer, for $j \geq 1$) and with $\widehat{p}(0) \leq (p(0))^\alpha$, for some $\alpha > 0$, produces a sequence of polynomials $p = p^{(0)}, p^{(1)}, p^{(2)}, \ldots$, such that $\deg p^{(n)} \to \infty$, $p^{(n)}(0) \to \infty$ and

$$(p^{(n)})\widehat{\ }(0) \leq (p^{(n)}(0))^\alpha$$

This shows $M(s) \leq C s^{1/\alpha}$, with C dependent on the initial p only.

2 Bounds on random trigonometric polynomials

In [7] the following classical theorem was used to estimate the size of a random polynomial

Theorem 1 (**Salem and Zygmund** [9], [5, p. 69]) *Let $f_1(x), \ldots, f_n(x)$, be trigonometric polynomials of degree at most m, and $\xi_1, \ldots, \xi_n$ be independent random variables, which satisfy $\mathbf{E} e^{\lambda \xi_j} \leq e^{\lambda^2/2}$, for all j and $\lambda > 0$. (subnormal random variables). Write*

$$f(x) = \sum_{j=1}^{n} \xi_j f_j(x)$$

Then, for some $C > 0$,

$$\Pr\left(\|f\|_\infty \geq C \left(\sum_{j=1}^{n} \|f_j\|_\infty{}^2 \log m \right)^{1/2} \right) \leq \frac{1}{m^2}$$

Theorem 1 was used in [7] to change the coefficients of a polynomial to integers without a big loss:

[1]After this paper was submitted for publication the author learned that the method employed for the proof of the basic result has also appeared in [6].

Corollary 1 *Let $p(x) = p_0 + \sum_{j=1}^N p_j \cos jx$ and define the random polynomial $r(x)$ so that $p(x)+r(x)$ has always integral coefficients (except perhaps the constant coefficient):*

$$r(x) = \sum_{j=1}^N \xi_j \cos jx$$

with $\xi_j = 0$ if p_j is an integer, else

$$\xi_j = \begin{cases} \lfloor p_j \rfloor - p_j & \text{with probability } \lceil p_j \rceil - p_j \\ \lceil p_j \rceil - p_j & \text{with probability } p_j - \lfloor p_j \rfloor \end{cases}$$

Then $\Pr\left(\|r\|_\infty \ll (N \log N)^{1/2}\right) \to 1$, as $N \to \infty$.

Proof of Corollary 1: The above defined ξ_j are subnormal (see for example [1, p. 235, Lemma A.6]). Theorem 1 can now be applied.

The following theorem of Spencer [10] is sometimes better than the Salem-Zygmund theorem, though unfortunately only in the symmetric case $\xi_j = \pm 1$ (Rademacher random variables).

Theorem 2 (**Spencer** [10]) *Let a_{ij}, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$ be such that $|a_{ij}| \le 1$. Then there are signs $\epsilon_1, \ldots, \epsilon_{n_2} \in \{-1, 1\}$ such that, for all i,*

$$|\sum_{j=1}^{n_2} \epsilon_j a_{ij}| \le C n_1^{1/2} \tag{6}$$

Notice there is no dependence of the bound on n_2.

Corollary 2 *Let $f_1(x), \ldots, f_n(x)$, $\|f_j\|_\infty \le C$, be trigonometric polynomials of degree at most m. Then there is a choice of signs $\epsilon_1, \ldots, \epsilon_n \in \{-1, 1\}$ such that*

$$\|\sum_{j=1}^n \epsilon_j f_j\|_\infty \le C m^{1/2}$$

Proof of Corollary 2: For $i = 1, \ldots, 10m$, $j = 1, \ldots, n$ define $a_{ij} = f_j(x_i)$, where $x_i = i\frac{2\pi}{10m}$. Let $\epsilon_1, \ldots, \epsilon_n$ be the sequence of signs given by Theorem 2 for the matrix a_{ij} and write $f = \sum_{j=1}^n \epsilon_j f_j$. There is $x_0 \in [0, 2\pi]$ such that $|f(x_0)| = \|f\|_\infty$. For some k we have $|x_k - x_0| \le \frac{2\pi}{10m}$. By Bernstein's inequality, $\|f'\|_\infty \le m\|f\|_\infty$, we get

$$|f(x_0) - f(x_k)| \le \frac{2\pi}{10m}\|f'\|_\infty \le \frac{2\pi}{10}|f(x_0)|$$

which, since $\frac{2\pi}{10} < 1$, implies

$$\|f\|_\infty = |f(x_0)| \le C|f(x_k)| = C|\sum_{j=1}^n \epsilon_j a_{kj}| \le C m^{1/2}$$

and the proof is complete.

Corollary 2 is better than Theorem 1 only when $m/\log m = o(n)$. It is a strictly symmetric result and cannot directly be applied to modify a polynomial $p(x)$ so that it has integral coefficients, as we need to do in our case. We show in Section 3 that a sequence of applications of Corollary 2 is needed.

3 Proof of the inequality $M(s) \ll s^{1/3}$

Since Corollary 2 only allows us to choose random signs, we cannot use it directly (as we used the Salem-Zygmund theorem) to modify the coefficients of a polynomial to integers, while controlling the size of the change. In this section we show how to modify the coefficients little by little, to achieve the same result.

Let $\alpha > 0$ and define

$$a(x) = \alpha K_A(x) = \sum_{j=0}^{A} a_j \cos jx$$

Suppose $\epsilon > 0$ and the non-negative integer k_0 is such that for some non-negative integers b_j

$$|a_j - b_j 2^{-k_0}| \le \epsilon \text{ for all } j = 1, \ldots, A$$

We shall define a finite sequence of polynomials

$$a^{(0)}(x) = a_0 + \sum_{j=1}^{A} b_j 2^{-k_0} \cos jx,\ a^{(1)}(x),\ \ldots\ ,\ a^{(k_0)}(x)$$

inductively, so that if $a^{(k)}(x) = a_0 + \sum_{j=1}^{A} a_j^{(k)} \cos jx$ then, for each $j = 1 \ldots A$,

$$a_j^{(k)} = b_j^{(k)} 2^{k-k_0} \tag{7}$$

for some non-negative integers $b_j^{(k)}$. We define inductively the coefficients of $a^{(k+1)}$ as follows. If $b_j^{(k)}$, $j > 0$, is even then $a_j^{(k+1)} = a_j^{(k)}$. Else define

$$a_j^{(k+1)} = a_j^{(k)} + \epsilon_j^{(k)} 2^{k-k_0} \tag{8}$$

where $\epsilon_j^{(k)} \in \{-1, 1\}$ are such that

$$\|\sum_{b_j^{(k)} \text{odd}} \epsilon_j^{(k)} \cos jx\|_\infty \le C A^{1/2} \tag{9}$$

The existence of the signs $\epsilon_j^{(k)}$ is guaranteed by Corollary 2. Notice that (8) implies the preservation of (7) by the inductive definition. We deduce from (9) that

$$\|a^{(k+1)} - a^{(k)}\|_\infty \le C 2^{k-k_0} A^{1/2} \tag{10}$$

The polynomial $a^{(k_0)}$ has integral coefficients (except perhaps for the constant coefficient). Summing (10) we get

$$\|a - a^{(k_0)}\|_\infty \le \|a - a^{(0)}\|_\infty + \|a^{(0)} - a^{(k_0)}\|_\infty \le A\epsilon + C A^{1/2}$$

Choose $\epsilon = 1/A$ to get $\|a - a^{(k_0)}\|_\infty \le C A^{1/2}$. On the other hand, the coefficients of a and $a^{(k_0)}$ differ by at most 1 and this implies that for the non-negative polynomial $p(x) = a^{(k_0)}(x) + \|a - a^{(k_0)}\|_\infty$ we have

$$p(0) \ \ge\ a(0) - A \ge C\alpha A - A \tag{11}$$

$$\widehat{p}(0) \ =\ \alpha + \|a - a^{(k_0)}\|_\infty \le \alpha + C A^{1/2} \tag{12}$$

Select $\alpha = A^{1/2}$ to get $\widehat{p}(0) \ll A^{1/2}$ and $p(0) \gg A^{3/2}$. Since p has integral coefficients, we have exhibited a polynomial that achieves $M(s) \ll s^{1/3}$, and the proof is complete.

Remark on cosine sums

Applying the method of the preceding proof on the coefficients of the Fejér kernel $K_A(x)$, one ends up with a non-negative polynomial of degree at most A, which is of the form

$$p(x) = p_0 + 2\sum_{j=1}^{k} \cos \lambda_j x$$

where $\lambda_j \in \{1, \ldots, A\}$ are distinct. We have $\|K_A - p\|_\infty \ll A^{1/2}$ which, since $p_0 = \frac{1}{2\pi}\int_0^{2\pi} p(x)dx$, implies

$$p_0 \ll A^{1/2} \text{ and } p(0) \gg A$$

Thus p is a new example of a cosine sum that achieves the upper bound in (1). It is not as simple as the one mentioned in the introduction but the spectrum of it is much denser: $\frac{1}{2}A + O(A^{1/2})$ cosines with frequencies from 1 to A.

Since the Dirichlet kernel

$$D_A(x) = \sum_{j=-A}^{A} e^{ijx} \tag{13}$$

$$= 1 + 2\sum_{j=1}^{A} \cos jx \tag{14}$$

$$= \frac{\sin\,(A+\frac{1}{2})x}{\sin \frac{x}{2}} \tag{15}$$

has a minimum asymptotically equal to $-\frac{4}{3\pi}A$, it is conceivable that one may be able to raise the above number of cosine from $\frac{1}{2}A + O(A^{1/2})$ to

$$(1 - \frac{4}{6\pi})A + o(A)$$

In other words, since

$$\min_x \sum_{j=1}^{A} \cos jx = -\frac{4}{6\pi}A + o(A)$$

one must remove at least $\frac{4}{6\pi}A$ cosines from the above sum, in order to make its minimum be $o(A)$, in absolute value.

4 The construction

Suppose we are given a polynomial $p(x) \geq 0$ of degree d, whose non-constant coefficients are even non-negative integers, which satisfies

$$\widehat{p}(0) \leq (p(0))^\alpha$$

for some $\alpha > 0$. Define the infinite sequence of non-negative polynomials $p = p^{(1)}, p^{(2)}, p^{(3)}, \ldots,$ with the recursive formula

$$p^{(k+1)}(x) = p^{(k)}((d+1)x) \cdot p(x) \tag{16}$$

Since p has even non-constant coefficients, the Fourier coefficients of all $p^{(k)}$ are non-negative integers. The spectrum of the first factor in (16) is supported by the multiples of $d+1$, and that of the second factor is supported by the interval $[-d,d]$. This implies that $(p^{(k+1)})\widehat{\ }(0) = (p^{(k)})\widehat{\ }(0)\widehat{p}(0)$. We obviously have $p^{(k+1)}(0) = p^{(k)}(0)p(0)$. We conclude that for all $k \geq 0$

$$(p^{(k)})\widehat{\ }(0) = (\widehat{p}(0))^k \text{ and } p^{(k)}(0) = (p(0))^k$$

and consequently

$$(p^{(k)})\widehat{\ }(0) \leq (p^{(k)}(0))^\alpha$$

So, if s is a power of $p(0)$, we have $M(s) \leq s^\alpha$, and for any s we have $M(s) \leq Cs^\alpha$, where $C = (p(0))^\alpha$.

As an example we give

$$p(x) = 4 + 4\cos x + 2\sum_{j=2}^{10} \cos jx$$

which can be checked numerically to be positive and has constant coefficient $\widehat{p}(0) = 4$ and $p(0) = 26$. This gives $\alpha = \log 4/\log 26 = .42549\cdots$.

In view of the above construction, finding a single polynomial p with $\widehat{p}(0) \leq (p(0))^\alpha$, with $\alpha < 1/3$, will prove that the result in this paper is not the best possible. The above example was actually found by a computer but if no more insight is gained into how these good "seed" polynomials look like, the computing time grows dramatically as we increase the degree of the polynomial.

Acknowledgement

I would like to thank P. J. Cohen and A. M. Odlyzko for their help.

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