Distance sets corresponding to convex bodies

Mihail N. Kolountzakis*

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Abstract

Suppose that $K \subseteq \mathbb{R}^d$, $d \geq 2$, is a 0-symmetric convex body which defines the usual norm

$$\|x\|_K = \sup\{t \geq 0 : \quad x \notin tK\}$$

on $\mathbb{R}^d$. Let also $A \subseteq \mathbb{R}^d$ be a measurable set of positive upper density ρ. We show that if the body K is not a polytope, or if it is a polytope with many faces (depending on ρ), then the distance set

$$D_K(A) = \{\|x-y\|_K : \quad x, y \in A\}$$

contains all points $t \geq t_0$ for some positive number t_0. This was proved by Furstenberg, Katznelson and Weiss, by Falconer and Marstrand and by Bourgain in the case where K is the Euclidean ball in any dimension greater than 1. As corollaries we obtain (a) an extension to any dimension of a theorem of Iosevich and Łaba regarding distance sets with respect to convex bodies of well-distributed sets in the plane, and also (b) a new proof of a theorem of Iosevich, Katz and Tao about the nonexistence of Fourier spectra for smooth convex bodies with positive curvature.

1 Introduction

Suppose that $K \subseteq \mathbb{R}^d$, $d \geq 2$, is a 0-symmetric convex body which defines the usual norm

$$\|x\|_K = \sup\{t \geq 0 : \quad x \notin tK\}$$

on $\mathbb{R}^d$. Define the K-distance set of $A \subseteq \mathbb{R}^d$ as the set of K-distances that show up in A:

$$\begin{aligned} D_K(A) &= \{\|x-y\|_K : \quad x, y \in A\} && (1)\\ &= \{t > 0 : \ \exists x, y \in A : \ y - x \in t \cdot \partial K\}. && (2) \end{aligned}$$

Furstenberg, Katznelson and Weiss [3], Falconer and Marstrand [2] (in the plane) and Bourgain [1] have proved that if B is the Euclidean ball and A has positive upper density, i.e., if there is $\rho > 0$ such that there are arbitrarily large cubes Q in which A has a fraction at least ρ of their measure: $|A \cap Q| \geq \rho |Q|$, then $D_B(A)$ contains all numbers $t \geq t_0$, for some t_0. In this paper we study the following question: for which convex bodies K in place of B is this true? We obtain that if this property (of eventually all numbers showing up in $D_K(A)$) fails for some set A with

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positive upper density then the body K is necessarily a polytope, and with a number of faces that is bounded above by a number that is inversely proportional to the density of A.

To state our results more precisely let us call a finite measure $\mu \in M(\mathbb{R}^d)$ *ϵ-good* if for some $R > 0$ we have $|\widehat{\mu}(x)| \le \epsilon$ for $|x| \ge R$. In what follows the Fourier Transform of a measure is defined by

$$\widehat{\mu}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle x, \xi \rangle} \, d\mu(x).$$

Theorem 1 *Suppose that $A \subseteq \mathbb{R}^d$, $d \ge 2$, has upper density equal to $\epsilon > 0$ and that the 0-symmetric convex body K affords $(C_d \epsilon)$-good probability measures supported on its boundary (the constant C_d depends on the dimension only). Then $D_K(A)$ contains all positive real numbers beyond a point.*

This is complemented by the following result.

Theorem 2 *Suppose K is a 0-symmetric convex body. Then*

(a) If K is a polytope with N non-parallel face directions then it does not afford ϵ-good probability measures on its boundary if $\epsilon < 1/(\sqrt{2}N)$.

(b) If K is a polytope with N non-parallel face directions then it affords $(\frac{1}{N} + \delta)$-good probability measures on its boundary for all $\delta > 0$.

If K is not a polytope then it affords ϵ-good probability measures on its boundary for all $\epsilon > 0$.

Remark 1 *It is not important that K is a* symmetric *body. One can express the difference set $D_K(A)$ using* (2) *to avoid speaking of the K-norm in Theorem 1, which holds for non-symmetric bodies as stated, and Theorem 2 is also essentially correct.*

Theorems 1 and 2 allow us to prove the following result, a weaker version of which was the motivation for this work and was proved in dimension 2 by Iosevich and Łaba [5]. A set $\Lambda \subseteq \mathbb{R}^d$ is called *well-distributed* if there is a constant $r > 0$ such that every cube of side r contains at least one point of Λ. And a set $D \in \mathbb{R}$ is called *separated* if there exists $\epsilon > 0$ such that $|x - y| \ge \epsilon$ for any two distinct $x, y \in D$.

Corollary 1 *Suppose that Λ is a well-distributed subset of $\mathbb{R}^d$ and K is a 0-symmetric convex body. If $D_K(\Lambda) \cap \mathbb{R}^+$ has infinitely many gaps of length at least $\epsilon > 0$ then K is a polytope.*

Remark 2 *It was proved in [5] that if Λ is a well-distributed set in the plane and $D_K(\Lambda)$ is separated then K is a polygon.*

Remark 3 *By taking K to be the unit ball of ℓ_1^d and $\Lambda = \mathbb{Z}^d$ we see that there are indeed polytopes and well-distributed sets for which $D_K(\Lambda)$ is separated. In this case $D_K(\Lambda) \subseteq \{0, 1, 2, \ldots\}$. It is not clear which polytopes and sets Λ can play this role.*

Proof of Corollary 1. Suppose that the intervals $(x_k, x_k + \epsilon)$, $x_k \to +\infty$, do not intersect $D_K(\Lambda)$. Let A be the subset of $\mathbb{R}^d$ that arises if we put a copy of the body $(\epsilon/10)K$ centered at each point of Λ. By the fact that Λ is well-distributed we obtain that A has positive upper density and for any two points $x, y \in A$, we can write $x = \lambda + x_1$, $y = \mu + y_1$, with $\lambda, \mu \in \Lambda$, $x_1, y_1 \in (\epsilon/10)K$. It follows that $x - y = (\lambda - \mu) + (x_1 - y_1)$ and

$$\frac{1}{5}\epsilon \geq \|x_1 - y_1\|_K \geq |\ \|x - y\|_K - \|\lambda - \mu\|_K\ |,$$

which implies that $D_K(A)$ does not intersect the intervals $(x_k + \epsilon/5, x_k + 4\epsilon/5)$, $k = 1, 2, \ldots$, hence $D_K(A)$ cannot contain a half-line. From Theorems 1 and 2 it follows that K is a polytope, and with a number of faces which is bounded above by a function of the upper density of A.
□

From Corollary 1 one can show that smooth convex bodies with everywhere positive curvature do not have Fourier spectra, a fact first proved by Iosevich, Katz and Tao [4], who used a different approach. A bounded open domain $\Omega \subseteq \mathbb{R}^d$ is said to have the set $\Lambda \subseteq \mathbb{R}^d$ as a *Fourier spectrum* if the collection of exponentials

$$E(\Lambda) = \{\exp(2\pi i \langle \lambda, x \rangle) :\ \lambda \in \Lambda\}$$

is an orthogonal basis for $L^2(\Omega)$. It is easy to see from the orthogonality that any two distinct points λ and μ of a Fourier spectrum must satisfy

$$\widehat{\chi_\Omega}(\lambda - \mu) = 0. \tag{3}$$

One can also see (see e.g. [6]) that Λ is a spectrum of Ω if and only if

$$\sum_{\lambda \in \Lambda} |\widehat{\chi_\Omega}(x - \lambda)|^2 = |\Omega|^2,$$

for almost every $x \in \mathbb{R}^d$. From this "tiling" property of the integrable function $|\widehat{\chi_\Omega}|^2$ on deduces that Λ is well-distributed. In fact it has density equal to $|\Omega|$.

Corollary 2 (Iosevich, Katz and Tao [4])
If K is a smooth, 0-symmetric convex body with everywhere positive Gaussian curvature it does not have a Fourier spectrum.

Remark 4 *In [4] the result is proved without assuming positive curvature but only smoothness. We can prove this stronger result with our method and the method is essentially what is given below but without using the surface measure in our use of Theorem 1. Instead we use a smooth measure supported in a neighborhood of some point of positive curvature. We will make the details clear in a later paper.*

Proof of Corollary 2. Suppose Λ is a Fourier spectrum of K. It is a well known fact (see, for example, [4]) that if ξ is a zero of $\widehat{\chi_K}$ and $\xi \to \infty$ then

$$\|\xi\|_{K^o} = \left(\frac{\pi}{2} + \frac{d\pi}{4}\right) + k\pi + o(1), \quad (\xi \to \infty),$$

where K^o is the dual body (which is also smooth), d is the dimension and k is an integer.

Let $R > 0$ be such that any zero ξ of $\widehat{\chi_K}$, outside a cube of side R centered at the origin, satisfies

$$\|\xi\|_{K^o} = \left(\frac{\pi}{2} + \frac{d\pi}{4}\right) + k\pi + \theta, \quad (k \in \mathbb{Z},\ |\theta| < \pi/10).$$

We also take R to be large enough so as to be certain that we find at least one Λ-point in any cube of side R. We can do this since Λ is well-distributed.

Let now the set Λ' arise by keeping only one point of Λ in each cube of the type $Rn + (-R/2, R/2)^d$, with $n \in \mathbb{Z}^d$ having all its coordinates even. We keep nothing outside these cubes. It follows that Λ' is also a well distributed set and that for any two distinct points λ and μ of Λ', μ is not contained in the cube of side R centered at λ. From (3) we obtain that for any two distinct points $\lambda, \mu \in \Lambda'$ we have

$$\|\lambda - \mu\|_{K^o} = k\pi + \theta, \quad (k \in \mathbb{Z},\ |\theta| \leq \pi/5).$$

This means that the set of K^o distances $D_{K^o}(\Lambda')$ has infinitely many gaps of length at least $3\pi/5$, so by Corollary 1 K^o should be a polytope, which is a contradiction.
□

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2 Proofs of the main theorems

Proof of Theorem 1.

Notation: $B_1(0)$ is the unit ball in $\mathbb{R}^d$ and ω_d is its volume. In what follows the dimension $d \geq 2$ is fixed and constants may depend on it.

As in [1], whose method we follow, it suffices to prove the following theorem.

Theorem A Suppose $\epsilon, R > 0$ and let K be a 0-symmetric convex body contained in $B_1(0)$ and σ be a probability measure on ∂K such that

$$|\widehat{\sigma}(\xi)| \leq \eta(\epsilon) := \frac{\omega_d}{80 \cdot 4^d \pi^d}\epsilon, \quad \text{if } |\xi| \geq R. \tag{4}$$

Then there is $J = J(\epsilon, R)$ such that if $A \subseteq B_1(0)$ is a set of measure ϵ and $0 < t_j < 1$, $j \geq 1$, is a sequence with $t_{j+1} < \frac{1}{2}t_j$ then there is $j \leq J$ such that $t_j \in D_K(A)$.

Theorem A implies Theorem 1: Suppose that K is a 0-symmetric convex body, and $\sigma \in M(\partial K)$ is a probability measure on its surface which satisfies $|\widehat{\sigma}(\xi)| \leq \eta(\epsilon)$ if $|\xi| \geq R$, and let $A \subseteq \mathbb{R}^d$ be a measurable set with upper density larger than $\epsilon > 0$. Suppose also that Theorem 1 fails and there is a sequence $0 < x_1 < x_2 < \cdots$ tending to infinity all of whose elements are not in $D_K(A)$.

Renaming, we pass to a lacunary subsequence of x_j such that $x_{j+1} > 2x_j$ with $J = J(C_d\epsilon, R)$ terms. Let Q be a cube of side-length greater than $10x_J$ for which $|Q \cap A| \geq \epsilon|Q|$. Scaling everything down to $B_1(0)$ and renaming the scaled x_j as t_j (and reversing their order so as to have a decreasing sequence) we now have a set of measure $\geq C\epsilon$ contained in $B_1(0)$ and a sequence $t_1 > t_2 > \cdots > t_J$, with $t_{j+1} < \frac{1}{2}t_j$.

By Theorem A there exists a $j \leq J$ such that t_j appears as a K-distance in the scaled-down set A, which implies that the corresponding $x_{j'}$ appears as a K-distance in the set A, a contradiction.
□

Proof of Theorem A. It is enough to show that there is $j \leq J(\epsilon, R)$ such that, for $t = t_j$,

$$\int\int f(x)f(x+ty)dxd\sigma(y) > 0,$$

where f is the indicator function of A. The integral on the left may be rewritten as a constant $C(t)$ times

$$\int \left|\widehat{f}(\xi)\right|^2 \widehat{\sigma}(t\xi)\ d\xi. \tag{5}$$

The integral in (5) is broken into three parts

$$I_1 = \int_{|\xi|\le\frac{\delta}{t}} \cdot, \quad I_2 = \int_{\frac{\delta}{t}<|\xi|<\frac{1}{\delta t}} \cdot, \quad I_3 = \int_{\frac{1}{\delta t}\le|\xi|} \cdot.$$

where δ will be chosen later to be $1/R$.

We shall now need three lemmas to control the quantities I_1, I_3, and I_2. I_1 is the main positive term and I_2, I_3 will be shown to be small for some t_j with $j \le J(\epsilon, R)$.

Lemma 1 *Let K be a 0-symmetric convex body contained in the unit ball $B_1(0)$. Let also σ be a probability measure on ∂K and $A \subseteq B_1(0)$ be a measurable set with indicator function f. Then, writing*

$$I_1(t,\delta) = \int_{|\xi|\le\frac{\delta}{t}} \left|\widehat{f}(\xi)\right|^2 \widehat{\sigma}(t\xi)\ d\xi,$$

we have

$$I_1(t,\delta) \ge \frac{\omega_d}{8\cdot 4^d\pi^d}|A|^2, \qquad (\textit{if } t \le 4\pi\delta < 1). \tag{6}$$

Proof of Lemma 1. Since f and σ are supported in $B_1(0)$ it follows that $\left|\nabla_u \widehat{f}\right| \le 2\pi|A|$ and $|\nabla_u \widehat{\sigma}| \le 2\pi$, for all directions $u \in S^{d-1}$. Hence

$$\widehat{f}(y) \ge \frac{|A|}{2}, \qquad \text{if } |y| \le \tfrac{1}{4\pi},$$

and

$$\widehat{\sigma}(y) \ge \frac{1}{2}, \qquad \text{if } |y| \le \tfrac{1}{4\pi}.$$

So, since $|t\xi| \le \delta \le \frac{1}{4\pi}$ in the range of integration,

$$\begin{aligned} I_1(t,\delta) &\ge \int_{|\xi|\le\frac{1}{4\pi}} \left|\widehat{f}(\xi)\right|^2 \widehat{\sigma}(t\xi)\ d\xi \\ &\ge \int_{|\xi|\le\frac{1}{4\pi}} \left(\frac{|A|}{2}\right)^2 \frac{1}{2}\ d\xi \\ &\ge \frac{1}{8}|A|^2 \left|\left\{|\xi| \le \frac{1}{4\pi}\right\}\right| \\ &\ge \frac{\omega_d}{8\cdot 4^d\pi^d}|A|^2. \end{aligned}$$

□

Lemma 2 *Suppose that σ is any measure for which $|\widehat{\sigma}(x)| \le \eta$ provided $|x| \ge R$ and write*

$$I_3(t,\delta) = \int_{|\xi|\ge\frac{1}{\delta t}} |g(\xi)|^2 \widehat{\sigma}(t\xi)\ d\xi,$$

where $g \in L^2(\mathbb{R}^d)$*. Then*

$$|I_3(t,\delta)| \le \eta \int |g|^2, \qquad (\textit{if } \delta \le \tfrac{1}{R}). \tag{7}$$

Proof of Lemma 2.

$$\begin{aligned} |I_3(t,\delta)| &\le \eta \int_{|\xi| \ge \frac{R}{t}} |g(\xi)|^2 \, d\xi \\ &\le \eta \int |g|^2. \end{aligned}$$

□

Lemma 3 *If* $t, \delta, \theta > 0$*,* $\delta < 1$*,* σ *is a finite measure with total variation at most* 1*,* $f \in L^2(\mathbb{R}^d)$ *with* $\epsilon = \int |f|^2$*,*

$$I_2(t,\delta) = \int_{\frac{\delta}{t} < |\xi| < \frac{1}{\delta t}} \left| \widehat{f}(\xi) \right|^2 \widehat{\sigma}(t\xi) \, d\xi,$$

and $\{t_j\}_{j=1}^{\infty}$ *is a sequence with*

$$0 < t_j < 1 \text{ and } t_{j+1} < \frac{1}{2} t_j$$

then there is an index

$$j \le \frac{2}{\log 2} \, \theta^{-1} \frac{1}{\epsilon} \log \frac{1}{\delta}$$

such that

$$|I_2(t,\delta)| \le \theta \epsilon^2. \tag{8}$$

Proof of Lemma 3. Note first that $\log \frac{1}{t_{j+1}} - \log \frac{1}{t_j} \ge \log 2$. For $x \in \mathbb{R}$ let $N(x)$ be the number of intervals $\left(\frac{\delta}{t_j}, \frac{1}{\delta t_j} \right)$ to which x belongs. It follows that

$$N(x) \le \frac{2}{\log 2} \, \log \frac{1}{\delta}$$

as $x \in \left(\frac{\delta}{t_j}, \frac{1}{\delta t_j} \right)$ is equivalent to

$$\log \frac{1}{t_j} - \log \frac{1}{\delta} < \log x < \log \frac{1}{t_j} + \log \frac{1}{\delta}.$$

For any positive integer J we have thus

$$\sum_{j=1}^{J} |I_2(t_j,\delta)| \le \int_{\mathbb{R}} \left| \widehat{f}(\xi) \right|^2 \widehat{\sigma}(t\xi) N(|\xi|) \, d\xi \le \frac{2}{\log 2} \, \log \frac{1}{\delta} \cdot \epsilon,$$

by $\int \left| \widehat{f}(\xi) \right|^2 d\xi = \epsilon$ and $|\widehat{\sigma}(\xi)| \le 1$. Hence, if we let

$$J = \frac{2}{\log 2} \, \theta^{-1} \frac{1}{\epsilon} \log \frac{1}{\delta},$$

we obtain that there is $j \le J$ for which

$$|I_2(t_j, \delta)| \le \theta\epsilon^2.$$

$\square$

Proof of Theorem A (continued). We may assume $R \ge 4\pi$. Let $\delta = 1/R$ and assume $t \le 4\pi\delta$. By Lemma 1 we get

$$I_1(t, \delta) \ge \frac{\omega_d}{8 \cdot 4^d \pi^d}\epsilon^2. \tag{9}$$

By Lemma 2, applied to $g = \widehat{f}$ we get

$$|I_3(t, \delta)| \le \eta\epsilon = \frac{\omega_d}{80 \cdot 4^d \pi^d}\epsilon^2. \tag{10}$$

Inequalities (9) and (10) hold for all $t \le 4\pi/R$.

Define $j_0 = j_0(R)$ by $t_{j_0} \le 4\pi/R$ (clearly $j_0 \le C \log R$, as $t_1 \le 1$) and $\theta = \frac{\omega_d}{80 \cdot 4^d \pi^d}$, and apply Lemma 3 to the sequence $t_{j_0}, t_{j_0+1}, \ldots$. It follows that there is j with

$$j \le j_0 + 10\theta^{-1}\frac{1}{\epsilon}\log\frac{1}{\delta} \le C\frac{\log R}{\epsilon} =: J(\epsilon, R)$$

such that

$$|I_2(t_j, \delta)| \le \theta\epsilon^2. \tag{11}$$

Putting together (9), (10) and (11) we obtain for this j

$$\begin{aligned} I(t_j) &\ge I_1(t_j, \delta) - |I_2(t_j, \delta)| - |I_3(t_j, \delta)| \\ &\ge \frac{\omega_d}{8 \cdot 4^d \pi^d}\epsilon^2 - \frac{\omega_d}{80 \cdot 4^d \pi^d}\epsilon^2 - \frac{\omega_d}{80 \cdot 4^d \pi^d}\epsilon^2 \\ &\ge \frac{\omega_d}{40 \cdot 4^d \pi^d}\epsilon^2 \\ &> 0, \end{aligned}$$

which shows that $t_j \in D_K(A)$, as we had to show. The proof of Theorem A and therefore of Theorem 1 is complete.
$\square$

We denote by S^{d-1} the surface of the unit ball in $\mathbb{R}^d$ and whenever Ω is a hypersurface in $\mathbb{R}^d$ we denote by σ_Ω its surface measure. Also, if $x, y \in S^{d-1}$, by $\mathrm{dist}\,(x, y)$ we understand their geodesic distance in S^{d-1}, in other words the angle formed by x, 0 and y.

Lemma 4 *Let $\Theta \subseteq S^{d-1}$ satisfy $\Theta = -\Theta$, and $\delta > 0$. Suppose also that $K \subseteq \mathbb{R}^d$ is a convex body and $D \subseteq \partial K$ is a measurable subset of the boundary on which almost every point has a normal vector $\xi \in \Theta$. Let also σ_D be the surface measure of ∂K restricted to D. Then*

$$\lim_{t\to\infty} \widehat{\sigma_D}(t\eta) = 0, \tag{12}$$

uniformly for

$$\eta \in \mathcal{N} = \left\{x \in S^{d-1} : \ \ \mathrm{dist}\,(x, \Theta) \ge \delta\right\}.$$

Proof. If the set D is contained in a hyperplane then Θ may be taken to be a the set $\{\pm\theta_0\}$. If, in addition, it is a rectangle, one gets the validity of (12) by direct calculation of $\widehat{\sigma_D}$, which tends to 0 except in the normal direction θ_0 and does uniformly so if one keeps away from the normal direction by any fixed angle. If D is not a rectangle in its hyperplane one gets (12) by approximating the set D with a union of finitely many disjoint rectangles. Further, if D is polytopal, then it consists of a finite number of flat pieces D_i and we can represent σ_D as the sum of σ_{D_i}, the surface measure restricted to D_i. Since each of the $\widehat{\sigma_{D_i}}(t\eta)$ goes to 0 uniformly in $\mathcal{N}$ so does their sum $\widehat{\sigma_D}(t\eta)$.

To any measure $\mu \in M(\mathbb{R}^d)$ and direction $\eta \in S^{d-1}$ we associate the projection measure on the line of η, $\mu^\eta \in M(\mathbb{R})$, defined by

$$\mu^\eta(A) = \mu(A\eta + \eta^\perp), \quad (A \text{ a Borel subset of } \mathbb{R}), \tag{13}$$

where $\eta^\perp$ is the hyperplane orthogonal to η. By Fubini's theorem we see that

$$\widehat{\mu^\eta}(t) = \widehat{\mu}(t\eta), \quad t \in \mathbb{R}. \tag{14}$$

We now use the fact that for all $\epsilon, \delta > 0$ there exists a polytopal approximation P of D such that

- all normals to P are at distance $\le \delta/2$ from Θ,
- for all $\eta \in \mathcal{N}$ we have that the measures σ_D^η and σ_P^η are in fact L^1 functions and

$$\left\|\sigma_D^\eta - \sigma_P^\eta\right\|_{L^1(\mathbb{R})} \le \epsilon, \quad (\eta \in \mathcal{N}). \tag{15}$$

To prove the existence of such an approximation one first shows this under the assumption that the normal vector is a continuous function on D and then uses the fact that (see, for example, [7, p. 23]) we can throw away a part of a convex surface of arbitrarily small measure so that the normal is continuous on what remains.

From (15) and (14) it follows that $|\widehat{\sigma_D}(t\eta) - \widehat{\sigma_P}(t\eta)| \le \epsilon$ for all $t \in \mathbb{R}, \eta \in N$. Since ϵ is arbitrary Lemma 4 follows for the general convex surface piece D from the fact that it holds for polytopal surfaces.
□

Proof of Theorem 2. (a) Suppose that K is a symmetric polytope with N non-parallel faces and that μ is a probability measure on ∂K which is ϵ-good. We will show that $\epsilon \ge \frac{1}{\sqrt{2}N}$.

The distinct direction vectors $\pm\theta_1, \ldots, \pm\theta_N$ are all the normals that appear on the faces of K. Since all faces are partitioned into those which are normal to θ_1, normal to θ_2, and so on, it follows that at least one of these pairs of faces has total μ-measure at least $1/N$, say the pair of faces normal to θ_1.

The projection measure (see (13)) μ^{θ_1} has then one or two nonnegative point masses c_1 and c_2 of total mass at least $1/N$. But, if c_i, $i \in I$, are all the point masses of the finite measure μ^{θ_1}, Wiener's Theorem tells us that

$$\sum_{i \in I} |c_i|^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} \left|\widehat{\mu^{\theta_1}}(t)\right|^2 \, dt.$$

From (14) and the assumption that μ is ϵ-good it follows that the right hand side above is $\le \epsilon^2$, from which it follows that

$$\epsilon^2 \ge c_1^2 + c_2^2 \ge \frac{1}{2N^2},$$

since $c_1 + c_2 \ge 1/N$. This is the claimed inequality.

(b) For the remaining part of Theorem 2 we must show that whenever K is not a polytope or is a polytope with at least N non-parallel faces, we can find, for any $\delta > 0$, a probability measure on ∂K which is $(\frac{1}{N} + \delta)$-good. For this we recall that almost all points x on ∂K have a unique tangent hyperplane T_x, whose outward normal unit vector we denote by $n(x)$. This map $x \to n(x)$ is called the Gauss map and, through it, a measure is defined on S^{d-1}, called the area measure S_K:

$$S_K(A) = \sigma_{\partial K}\{x \in \partial K : \quad n(x) \in A\}, \quad \text{for any Borel set } A \subseteq S^{d-1}.$$

It is well known that the 0-symmetric convex body is a polytope with N pairs of opposite faces if and only if the measure S_K is a symmetric measure supported on N pairs of opposite points on the sphere S^{d-1}. Therefore the support of the area measure of K contains at least $2N$ points $\pm\theta_1, \ldots, \pm\theta_N$. For half of these points, the points $\theta_1, \ldots, \theta_N$, we choose an open neighborhood $N_i \subset S^{d-1}$, $i = 1, \ldots, n$, around each of them so that all these neighborhoods are disjoint, and we call $\delta_0 > 0$ the minimum geodesic distance between any two of them. Let then $D_i = n^{-1}(N_i) \subseteq \partial K$ be the parts of the boundary almost all points of which get mapped into N_i via the Gauss map $n(x)$. This implies that all points in D_i have a normal in N_i, and the D_i all have positive surface measure.

Define now the probability measure $\mu \in M(\partial K)$ to be an appropriate multiple of surface measure on each D_i, so as to have total mass $1/N$ on each D_i. Call μ_i the measure μ restricted to D_i. From Lemma 4 it follows that $\widehat{\mu_i}(t\eta) \to 0$ as $t \to \infty$ uniformly for all η which are distance at least $\delta_0/10$ from N_i. And for all x we have trivially $|\widehat{\mu_i}(x)| \le 1/N$.

Let now $\delta > 0$ and choose $R > 0$ large enough so that for all $i = 1, \ldots, N$ we have $|\widehat{\mu_i}(t\eta)| \le \delta/(N-1)$ if $|t| > R$ and η has distance more than $\delta_0/10$ from N_i. If now $x \in \mathbb{R}^d$ is an arbitrary vector with $|x| > R$ the vector $\eta = \frac{1}{|x|}x$ can have distance at most $\delta_0/10$ from at most one neighborhood N_i, say from N_1. If follows that

$$\begin{aligned}
|\widehat{\mu}(x)| &\le |\widehat{\mu_1}(x)| + \ldots + |\widehat{\mu_N}(x)| \\
&\le \frac{1}{N} + (N-1)\frac{\delta}{N-1} \\
&= \frac{1}{N} + \delta.
\end{aligned}$$

□

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Department of Mathematics, University of Crete, Knossos Ave., 714 09 Iraklio, Greece.
E-mail: `kolount@member.ams.org`