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On a problem of Erdős and Turán and some related results

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Abstract

We employ the probabilistic method to prove a stronger version of a result of Helm, related to a conjecture of Erdős and Turán about additive bases of the positive integers. We show that for a class of random sequences of positive integers A, which satisfy $|A \cap [1,x]| \gg \sqrt{x}$ with probability 1, all integers in any interval $[1,N]$ can be written in at least $c_1 \log N$ and at most $c_2 \log N$ ways as a difference of elements of $A \cap [1,N^2]$. We also prove several results related to another result of Helm. We show that for every sequence of positive integers M, with counting function $M(x)$, there is always another sequence of positive integers A such that $M \cap (A-A) = \emptyset$ and $A(x) > x/(M(x)+1)$. We also show that this result is essentially best possible, and we show how to construct a sequence A with $A(x) > cx/(M(x)+1)$ for which every element of M is represented exactly as many times as we wish as a difference of elements of A.

Mathematics Subject Classification: 11B13

Notation

All sequences we consider are sequences of distinct nonnegative integers. We write $\mathbb{N} = \{0,1,2,\ldots\}$. We denote by the lower case indexed letter the members of the sequence and by the capital letter the sequence as a set as well as its counting function. For example $A = \{a_0, a_1, a_2, \ldots\}$ denotes a sequence of distinct nonnegative integers and $A(x) = |A \cap [0,x]|$ denotes its counting function. The initial segment $A \cap [0,x]$ is denoted by $A^{\le x}$. The positive difference set $\{a-b:\ a,b \in A,\ a > b\}$ is denoted by $A-A$ and the sumset $\{a+b : a \in A,\ b \in B\}$ by $A+B$. We denote by C an arbitrary positive constant and we write $a \ll b$, if there exists a constant C such that $a \le Cb$. By $a \sim b$ or $a = (1+o(1))b$ we mean $\lim a/b = 1$ as a certain quantity, which will be clear from the context, approaches a limit. Similarly we write $a \lesssim b$ for $a \le (1+o(1))b$. We define several "representation" functions for a given set A:

$$\delta_A(x) = |\{(a,b) : a,b \in A,\ x = a-b\}|,$$

$$h_{A,N}(x) = \left|\{(a,b) : a,b \in A \cap [1,N^2],\ x = a-b\}\right|,$$

$$H_A(N) = \sum_{x=1}^{N} h_{A,N}(x),$$

and

$$r_A(x) = |\{(a,b) : a,b \in A,\ a \le b,\ x = a+b\}|.$$

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1 Introduction

A conjecture of Erdős and Turán [2] asserts that for any asymptotic basis (of order 2) of the positive integers, that is for any set $E \subseteq \mathbb{N}$ for which $r_E(x) > 0$ for all sufficiently large x, we have

$$\limsup_{x\to\infty} r_E(x) = \infty.$$

Erdős (c.f. [5]) has proved that it cannot be true that $r_E(x) = 1$ for all sufficiently large x, by showing that for any sequence E, with $E(x) \gg \sqrt{x}$, we have

$$H_E(N) \gg N \log N. \tag{1}$$

Indeed, any asymptotic basis E satisfies $E(x) \gg \sqrt{x}$ and if $r_E(x) = 1$ all sums we can form with two elements of E (with the exception of a finite number of elements of E) are distinct. This in turn implies that so are all the differences, that is $\delta_E(x) \le 1$ for all x, which makes (1) impossible.

Recently Helm [4] proved that (1) is best possible by explicitly constructing a sequence A, with $A(x) \gg \sqrt{x}$, for which

$$H_A(N) \ll N \log N. \tag{2}$$

Helm's proof does not provide any upper or lower bound on the individual $h_{A,N}(x)$ for $x \in [1, N]$, but only describes the average behaviour.

In addition to the above result Helm [4] constructed two sequences B and M, with $B(x) \gg \sqrt{x}$ and $M(x) \gg \log x$, for which $\delta_B(m_k) = 1$, for all k sufficiently large.

In this paper we improve both results of Helm.

We prove

Theorem 1 *Let a random sequence A be defined by letting $x \in A$ with probability $p_x = K/\sqrt{x}$ for $x \ge K^2$, $p_x = 0$ if $x < K^2$, for a constant K, independently for all x. Then, if the constant K is sufficiently large and with probability 1, there is an integer N_0 and positive constants c_1, c_2, c_3, c_4 such that*

$$c_1\sqrt{x} \le A(x) \le c_2\sqrt{x} \tag{3}$$

and

$$c_3 \log N \le h_{A,N}(m) \le c_4 \log N \tag{4}$$

for all $x, N \ge N_0$ and $1 \le m \le N$.

This implies the first result of Helm and with upper and lower estimates on the individual $h_{A,N}(m)$.

We also prove some results related to the second result of Helm mentioned above. Theorems 2–4 deal with the question of which sequences are avoidable by difference sets of dense sequences.

Theorem 2 *Let $M = \{m_0, m_1, \ldots\}$ be a sequence of positive integers. Then there is a sequence $A \subseteq \mathbb{N}$ such that $M \cap (A - A) = \emptyset$ and A is dense, that is:*

$$A(x) \ge \frac{x}{M(x)+1}, \text{ for all } x \in \mathbb{N}. \tag{5}$$

The proof of Theorem 2 is a straightforward construction. As an example, perhaps relevant to the Erdős-Turán conjecture, we see that if $M(x) \le \sqrt{x}$ then there is a sequence A, with $A(x) \ge \sqrt{x} - 1$, such that $M \cap (A - A) = \emptyset$.

The following result shows that Theorem 2 is essentially best possible.

Theorem 3 *Let $f(x) > 0$ be defined on $\mathbb{N}$ and assume that both $f(x)$ and $x/f(x)$ are non-decreasing and tend to infinity. Then there is a sequence of positive integers M, with $M(x) \lesssim x/f(x)$, such that for every sequence A, with $A(x) \ge f(x)$ for x sufficiently large, we have*

$$|M \cap (A - A)| = \infty.$$

That is for every lower bound – the function $f(x)$ – for the growth of $A(x)$ there is a not-very-dense sequence M that intersects infinitely often the difference set of every sequence A that meets the lower bound requirement. Again, in the case of quadratic growth we see that there is a sequence M, with $M(x) \lesssim \sqrt{x}$, which intersects infinitely often the difference set of *any* sequence A which satisfies $A(x) \ge \sqrt{x}$ for sufficiently large x.

Finally, we prove a result concerning the representation of the elements of a given sequence M as differences of elements from another sequence.

Theorem 4 *Let $M = \{m_0, m_1, \ldots\} \subseteq \mathbb{N}$ and assume $M(x) = o(x)$. Then there is $A = \{a_0, a_1, \ldots\} \subseteq \mathbb{N}$ such that $\delta_A(m_k) = 1$ for all k and*

$$A(x) \ge c\frac{x}{M(x)+1} \tag{6}$$

for all x, where c is a fixed positive constant.

The proof of Theorem 4 can also give us, for any given sequence $d_k \in \{0, 1, 2, \ldots, \infty\}$ (infinity included), a sequence A which satisfies the growth condition (6) and is such that $\delta_A(m_k) = d_k$ for all k.

2 Proofs

We need the following Lemma [1, p. 239] to bound the probability of large deviation of certain random variables.

Lemma 1 *If $Y = X_1 + \cdots + X_k$, and the X_j are independent indicator random variables, then for all $\epsilon > 0$*

$$\mathbf{Pr}\left[|Y - \mathbf{E}Y| > \epsilon\mathbf{E}Y\right] \le 2\exp(-c_\epsilon\mathbf{E}Y),$$

where $c_\epsilon > 0$ is a function of ϵ alone.

We call a random variable Y which, as above, is a *Sum of Indepenent Indicator Random Variables* a SIIRV.
Remark: Observe that if $Y = Y_1 + Y_2$, where Y_1 and Y_2 are SIIRV then we have

$$\mathbf{Pr}\left[|Y - \mathbf{E}Y| > \epsilon\mathbf{E}Y\right] \le 4\exp(-c_\epsilon\min\{\mathbf{E}Y_1, \mathbf{E}Y_2\}).$$

Proof of Theorem 1: Write $\chi_j = 1$ if $j \in A$, $\chi_j = 0$ otherwise, so that $\mathbf{E}\chi_j = p_j$. Notice that

$$\begin{aligned} A(x) &= \sum_{j=1}^{x} \chi_j, \\ h_{A,N}(m) &= \sum_{j=1}^{N^2-m} \chi_j\chi_{j+m} \end{aligned}$$

so that $A(x)$ is a SIIRV and $h_{A,N}(m)$ is the sum of two SIIRV:

$$h_{A,N}(m) = h^e_{A,N}(m) + h^o_{A,N}(m),$$

where

$$h^e_{A,N}(m) = \sum_{j=1}^{m}\sum_{k \text{ even}} \chi_{j+km}\chi_{j+(k+1)m}$$

and

$$h^o_{A,N}(m) = \sum_{j=1}^{m}\sum_{k \text{ odd}} \chi_{j+km}\chi_{j+(k+1)m}.$$

(We broke up $h_{A,N}(m)$ so that each χ_j appears at most once in each of the terms $h^e_{A,N}(m)$ and $h^o_{A,N}(m)$.) Then, as $x \to \infty$,

$$\begin{aligned}\mathbf{E}A(x) &= \sum_{j=1}^{x} p_j \sim \sum_{j=1}^{x} \frac{K}{\sqrt{j}} \\ &= K\sqrt{x}\sum_{j=1}^{x}\frac{1}{x}\frac{1}{\sqrt{j/x}} \\ &\sim 2K\sqrt{x},\end{aligned}$$

since $2 = \int_0^1 ds/\sqrt{s}$. We also have, for $m \le N$ and $N \to \infty$,

$$\begin{aligned}\mathbf{E}h_{A,N}(m) &= \sum_{j=1}^{N^2-m} p_j p_{j+m} \\ &\sim K^2 \sum_{j=1}^{N^2-m} \frac{1}{\sqrt{j(j+m)}} \\ &\le K^2 \sum_{j=1}^{N^2-m} \frac{1}{j} \\ &\sim 2K^2 \log N,\end{aligned}$$

and

$$\begin{aligned}\mathbf{E}h_{A,N}(m) &\gtrsim K^2 \sum_{j=1}^{N^2-m} \frac{1}{j+m} \\ &\gtrsim K^2 \log N.\end{aligned}$$

So we have

$$\mathbf{E}A(x) \sim 2K\sqrt{x} \tag{7}$$

as $x \to \infty$ and

$$K^2 \log N \lesssim \mathbf{E}h_{A,N}(m) \lesssim 2K^2 \log N \tag{8}$$

as $N \to \infty$, and for all $m \le N$. Notice that $\mathbf{E}h^e_{A,N}(m) \sim \frac{1}{2}\mathbf{E}h_{A,N}(m)$ and $\mathbf{E}h^o_{A,N}(m) \sim \frac{1}{2}\mathbf{E}h_{A,N}(m)$.

Now fix $\epsilon = 1/2$ and define the "bad" events

$$\begin{aligned}P_x &= \{|A(x) - \mathbf{E}A(x)| > \epsilon \mathbf{E}A(x)\}, \\ Q_{N,m} &= \{|h_{A,N}(m) - \mathbf{E}h_{A,N}(m)| > \epsilon \mathbf{E}h_{A,N}(m)\},\end{aligned}$$

for all x, N and $m \le N$. Using Lemma 1 and the remark following it we have

$$\mathbf{Pr}\left[P_x\right] \le 2\exp\left(-c_\epsilon \mathbf{E}A(x)\right) \le 2\exp\left(-\frac{1}{2}c_\epsilon 2K\sqrt{x}\right)$$

and

$$\mathbf{Pr}\left[Q_{N,m}\right] \le 4\exp\left(-\frac{1}{3}c_\epsilon \mathbf{E}h_{A,N}(m)\right) \le 4\exp\left(-\frac{1}{6}c_\epsilon K^2 \log N\right) = 4N^{-\frac{1}{6}c_\epsilon K^2}$$

for x and N sufficiently large. Thus

$$\sum_{x=1}^{\infty} \mathbf{Pr}\left[P_x\right] + \sum_{N=1}^{\infty}\sum_{m=1}^{N} \mathbf{Pr}\left[Q_{N,m}\right] \ll \sum_{x=1}^{\infty} \exp\left(-c_\epsilon K\sqrt{x}\right) + \sum_{N=1}^{\infty} N^{1-\frac{1}{6}c_\epsilon K^2}.$$

The first term in the right hand side is finite, and we choose K large enough to make the second term also finite, that is large enough to make $1-\frac{1}{6}c_\epsilon K^2 < -1$. Let now $\epsilon' \in (0,1)$ be arbitrary. Since the right hand side above is finite, we can find N_0 so that

$$\sum_{x\ge N_0} \mathbf{Pr}\,[P_x] + \sum_{N\ge N_0}\sum_{m=1}^{N} \mathbf{Pr}\,[Q_{N,m}] < \epsilon' \tag{9}$$

which means that, with probability at least $1-\epsilon'$, none of the events which appear in (9) holds. We conclude that, with probability at least $1-\epsilon'$,

$$K\sqrt{x} \lesssim A(x) \lesssim 3K\sqrt{x},$$

and

$$\frac{1}{2}K^2\log N \lesssim h_{A,N}(m) \lesssim 3K^2\log N,$$

for all $x, N \ge N_0$ and $1 \le m \le N$. Since ϵ' was arbitrary this concludes the proof. □

Proof of Theorem 2: We construct the sequence A with a "greedy" algorithm. Let $a_0 = 0$ and define inductively

$$a_{n+1} = \min\{y\in\mathbb{N}:\ y > a_n \ \&\ y \notin \{a_0,\ldots,a_n\} + M\}. \tag{10}$$

In words, we take a_{n+1} to be the least integer that does not destroy the desired property of the sequence A, namely that $\delta_A(m_k) = 0$ for all k. It is obvious that the set A defined by the above induction satisfies $M\cap(A-A)=\emptyset$.

We now bound from below the counting function of A. Assume that $y\in[0,x]\backslash A$. But then, by the way we construct A, there are a_k and m_l, both $\le x$, such that $y = a_k + m_l$. Thus

$$\left|[0,x]\backslash A\right| \le \left|\{a_k+m_l:\ a_k\le x,\ m_l\le x\}\right| \le A(x)M(x),$$

from which we conclude

$$A(x) = x+1-\left|[0,x]\backslash A\right| \ge x - A(x)M(x),$$

which proves the desired $A(x)\ge x/(M(x)+1)$. □

Proof of Theorem 3: For $s\in\mathbb{N}$ define $t=t(s)$ by

$$t = \min\{y\in\mathbb{N}:\ y>s \ \&\ f(y)>s\}$$

and the set M_s by

$$M_s = \{y\in\mathbb{N}:\ 0<y<t \ \&\ s|y\}.$$

Define the sequences $s_n, t_n\in\mathbb{N}$ inductively by $s_1 = 1$ and

$$s_{n+1} = \min\left\{y\in\mathbb{N}:\ y>t(s_n)\ \&\ \frac{y}{f(y)} \ge \left(\sum_{k=1}^{n}\frac{t_k}{f(t_k)}\right)^2\right\},$$

and by setting $t_n = t(s_n)$ for all n. Finally define $M=\bigcup_{n=1}^\infty M_{s_n}$.

We need to bound the counting functions of each M_{s_n}. We claim that for each $x\in\mathbb{N}$ we have $M_{s_n}(x)\le x/f(x)$. Indeed, if $x<t_n$ we have

$$M_{s_n}(x) = \left\lfloor\frac{x}{s_n}\right\rfloor \le \frac{x}{s_n} \le \frac{x}{f(x)},$$

because for this range of x we have $f(x)\le s_n$. On the other hand, if $x\ge t_n$ we have

$$M_{s_n}(x) = M_{s_n}(t_n-1) \le \frac{t_n-1}{f(t_n-1)} \le \frac{x}{f(x)},$$

since the integer t_n-1 is covered by the previous case and $x/f(x)$ is non-decreasing.

We now bound the counting function of M. Assume first that $s_{n+1} \le x < t_{n+1}$ for some $n \ge 0$. Then, by the previous calculation for $M_{s_n}(x)$,

$$\begin{aligned} M(x) &\le \sum_{k=1}^{n+1} M_{s_k}(x) \\ &= \sum_{k=1}^{n} M_{s_k}(t_k) + M_{s_{n+1}}(x) \\ &\le \sum_{k=1}^{n} \frac{t_k}{f(t_k)} + \frac{x}{f(x)} \\ &\le \left(\frac{x}{f(x)}\right)^{1/2} + \frac{x}{f(x)} \\ &\sim \frac{x}{f(x)}. \end{aligned}$$

If we have $t_n \le x < s_{n+1}$ for some $n \ge 1$ then we still have

$$M(x) = M(t_n - 1) \le \frac{t_n - 1}{f(t_n - 1)} \le \frac{x}{f(x)},$$

which completes the proof of $M(x) \lesssim x/f(x)$ for all x.

We still have to verify that $|M \cap (A - A)| = \infty$ for each sequence A of positive integers for which $A(x) \ge f(x)$ for all $x \ge x_0$. For this it suffices to show that $A - A$ intersects M in every $[s_n, t_n)$ interval, for large n. Look at n such that $s_n \ge x_0$. Since $A(t_n) \ge f(t_n) > s_n$ there exist two elements a, b of $A \cap (0, t_n]$, $a < b$, which are equal mod s_n. But then $s_n | b - a$ and $0 < b - a < t_n$ which implies that $b - a$ is in M_{s_n} and consequently in M, which we had to prove. □

Proof of Theorem 4: We construct A with a greedy algorithm which is a variation of the algorithm we used in the proof of Theorem 2. Loosely speaking, we construct a sequence A such that any new element we add does not create any new representations of any m_k as a difference from A. But occasionally we stop to add a pair of elements of the form $x, x + m_k$ to our set A so as to represent m_k once. What makes the construction work is that we are free to put off representing m_k until very late in the construction.

We need the following lemma.

Lemma 2 *Let M and $a_0, \ldots, a_n$ be given and be such that*

$$n \ge \alpha \frac{a_n}{M(a_n) + 1}.$$

Then we can extend $a_0, \ldots, a_n$ to an infinite sequence $A = \{a_0, \ldots, a_n, a_{n+1}, \ldots\}$ without adding any more representations of any elements of M, that is $\delta_A(m_k) = \delta_{\{a_0,\ldots,a_n\}}(m_k)$ for all k, and such that

$$A(x) \gtrsim \frac{x}{M(x) + 1}, \text{ as } x \to \infty, \tag{11}$$

and

$$A(x) \ge \frac{\alpha}{\alpha + 1} \frac{x}{M(x) + 1}, \text{ for all } x > a_n. \tag{12}$$

Proof of Lemma 2: For $k = n, n + 1, \ldots$, we define, as in the proof of Theorem 2,

$$a_{k+1} = \min\{y \in \mathbb{N} : \ y > a_k \ \& \ y \notin \{a_0, \ldots, a_k\} + M\}.$$

The sequence A thus constructed obviously adds no new representations of any m_k as a difference from A. For $x > a_n$ we have (with the same reasoning as in the proof of Theorem 2)

$$A(x) \ge x - a_n - A(x)M(x),$$

which implies

$$A(x) \geq \frac{x - a_n}{M(x) + 1}. \tag{13}$$

If $x \geq (\alpha + 1)a_n$ then (13) gives (12). If $a_n < x \leq (\alpha + 1)a_n$ then we have

$$A(x) \geq A(a_n) \geq \alpha \frac{a_n}{M(a_n) + 1} \geq \frac{\alpha}{\alpha + 1} \frac{x}{M(x) + 1},$$

which completes the proof of (12) for all x. The asymptotic inequality (11) is immediate from (13). □

We now proceed with the construction of the set A for Theorem 4. Let $\lambda \in (0, 1)$ be a fixed number (0.9 will do) and set $a_0 = k = 0$. We define the infinite set A by alternatingly applying the following two operations that take an initial segment of A and extend it.

1. If we have already defined $\{a_0, \ldots, a_n\}$ then we use Lemma 2 to define $a_{n+1}, \ldots, a_m$, such that

   $$(i)\ m > \lambda a_m / (M(a_m) + 1), \quad (ii)\ a_m > 100 m_k, \text{ and } (iii)\ M(a_m) > 100 m_k.$$

2. Having defined $\{a_0, \ldots, a_n\}$ we define the numbers a_{n+1} and a_{n+2} by

   $$\begin{aligned} a_{n+1} &= \min\{y \in \mathbb{N} : y > a_n \ \&\ y, y + m_k \notin \{a_0, \ldots, a_n\} + M\} \\ a_{n+2} &= a_{n+1} + m_k. \end{aligned}$$

   We then increment k by 1.

We apply operations 1 and 2 to the set A alternatingly, starting with operation 1.

Clearly the set A satisfies $\delta_A(m_k) = 1$ for all k, provided that it is infinite. We only have to verify that it satisfies the growth condition $A(x) \geq cx/(M(x) + 1)$, which, of course, implies that A is infinite, since $M(x) = o(x)$. It suffices to show that the inequality is satisfied for $x < N$ where N is the largest defined element of A at the end of each operation. We shall determine a value for the constant c at the end of the proof but we make no effort of getting the best value. (We believe that c can be arbitrarily close to 1.)

Analysis of operation 2: Operation 2 follows an application of operation 1, so we may assume that the elements $a_0, \ldots, a_n$ of A have been defined and satisfy conditions (i), (ii), and (iii) with n in place of m. We have to show that for all $x \in (a_n, a_{n+2}]$ we have the inequality

$$A(x) \geq c \frac{x}{M(x) + 1}.$$

Assume first that $x \in (a_n, a_{n+1})$. For each $y \in (a_n, x]$ we must either have $y \in A^{\leq a_n} + M^{\leq x}$ or $y + m_k \in A^{\leq a_n + m_k} + M^{\leq x + m_k}$. Since $A(a_n + m_k) \leq n + 2$, this implies that

$$x - a_n \leq 2(n + 2)M(x + m_k) \leq 2(n + 2)(M(x) + m_k) \leq 4nM(x),$$

from which we get

$$n \geq \frac{1}{4} \frac{x - a_n}{M(x) + 1}. \tag{14}$$

If $x \geq \mu a_n$ then $n \geq 1/4(1 - 1/\mu)x(M(x) + 1)^{-1}$ which implies

$$A(x) \geq \frac{1}{4}(1 - 1/\mu) \frac{x}{M(x) + 1}. \tag{15}$$

If, on the other hand, $a_n < x \leq \mu a_n$ then $A(x) \geq n \geq \lambda a_n (M(a_n) + 1)^{-1} \geq (\lambda/\mu)x(M(x) + 1)^{-1}$. For λ close to 1 and for $\mu = 2$ we get

$$A(x) \geq \min\{\lambda/2, 1/8\} \frac{x}{M(x) + 1} \geq \frac{1}{8} \frac{x}{M(x) + 1}$$

for all $x \in (a_n, a_{n+1})$. The remaining case $x \in [a_{n+1}, a_{n+2}]$ is easier. Since we have proved a lower bound for $x = a_{n+1} - 1$ we have

$$\begin{aligned} A(x) &\geq A(a_{n+1}-1) \geq \frac{1}{8}\frac{a_{n+1}-1}{M(a_{n+1}-1)+1} \\ &\geq \frac{1}{8}\frac{a_{n+1}-1}{M(x)+1} \\ &\geq \frac{1}{8}\frac{x-m_k-1}{M(x)+1} \\ &\geq \frac{1}{8}\frac{0.9\,x}{M(x)+1}. \end{aligned}$$

We have proved that for all $x \in (a_n, a_{n+2}]$ we have

$$A(x) \geq \frac{0.9}{8}\frac{x}{M(x)+1},$$

which completes the analysis of operation 2.

Analysis of operation 1: We only have to use Lemma 2 with $\alpha = 0.9/8$. We conclude that for all $x \in (a_n, a_m]$ we have

$$A(x) \geq \frac{0.9/8}{0.9/8+1}\,\frac{x}{M(x)+1}.$$

Thus we have proved Theorem 4 with $c = 0.9/8 \cdot (0.9/8+1)^{-1}$. □

Now suppose we want to achieve $\delta_A(m_k) = d_k$, where $d_k \in \mathbb{N} \cup \{\infty\}$ is a given sequence, and have $A(x)$ satisfy the same bound as in Theorem 4. Notice that in the previous proof we did not need the fact that the numbers m_k were distinct or non-decreasing.

All we have to do is construct a sequence $M' = \{m'_0, m'_1, \ldots\}$ in which each m_k appears exactly d_k times and apply our Theorem 4 to this sequence. We only need to assume $M(x) = o(x)$ as before, not that $M'(x) = o(x)$ (as a multiset).

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