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A new estimate for a problem of Steinhaus

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Abstract

Steinhaus asked whether there exists a subset of the plane which, no matter how translated and rotated, always contains exactly one point with integer coordinates. It is still unknown if there exist sets with the Steinhaus property.

Using harmonic analysis we prove that if a measurable set $S \subset \mathbf{R}^2$ satisfies $\int_S |x|^{\frac{10}{3}+\epsilon}\, dx < \infty$, for some $\epsilon > 0$, then it cannot have the Steinhaus property.

0. Introduction

0.1 A problem of Steinhaus. Steinhaus [8, problem 59] asked whether there is a planar set S which, no matter how translated and rotated, always contains exactly one point with integer coordinates.

Definition 1 *A set $S \subset \mathbf{R}^2$ has the Steinhaus property if for every $x \in \mathbf{R}^2$ and for every rotation $A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ we have*

$$\# \left(\mathbf{Z}^2 \cap (A_\theta S + x)\right) = 1, \tag{1}$$

where $A_\theta S + x = \{A_\theta s + x :\ s \in S\}$.

It is still unknown whether such a set exists.

0.2 Let $S \subset \mathbf{R}^2$ and denote by $\mathbf{1}_S$ its indicator function. Let also $A_\theta S$ denote the set S rotated by θ. The Steinhaus property is clearly equivalent to the statement:

$$\forall \theta \in [0, 2\pi),\ \forall x \in \mathbf{R}^2 :\ \sum_{n \in \mathbf{Z}^2} \mathbf{1}_{A_\theta S}(x - n) = 1. \tag{2}$$

Otherwise stated, a set S has the Steinhaus property if and only if it is a fundamental domain (i.e. contains exactly one element of each coset) of the groups $A_\theta \mathbf{Z}^2$ for all

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θ. Yet another way to say it is that $A_\theta S$ *tiles* the plane if translated at the locations $\mathbf{Z}^2$, and this is true for all θ.

0.3 In this paper we restrict ourselves to measurable $S \subset \mathbf{R}^2$ and we only demand that

$$\forall \theta \in \Theta, \text{ for almost every } x \in \mathbf{R}^2 : \sum_{n \in \mathbf{Z}^2} \mathbf{1}_{A_\theta S}(x - n) = 1, \qquad (3)$$

where $\Theta \subseteq [0, 2\pi)$ is dense in $[0, 2\pi)$, and "almost every" is taken in the sense of Lebesgue measure (the exceptional set of x's may depend on θ). It is known that several classes of sets S cannot have property (3), which we shall also call the Steinhaus property from now on. The spirit of the following results is to prove that the Steinhaus property is incompatible with fast decay of the set S outside of large disks or strips.

Sierpiński [9] first proved that a set which is bounded and either open or closed cannot have the Steinhaus property. Croft [2] and Beck [1] proved the same of any set which is bounded and measurable. (Several variations of the problem have been investigated by Komjáth [7] from a rather different point a view.) Croft's approach is more direct and geometric. Beck is using harmonic analysis. Beck's method was simplified and extended by the author [6]. In [6] it was proved that if the measure of the set

$$S \cap \{x \in \mathbf{R}^2 : |\langle x, u \rangle| > R\} \qquad (4)$$

decays like $\exp(-KR^2 \log^{1/2} R)$, for some large constant $K > 0$ and some unit vector $u \in \mathbf{R}^2$, then S cannot have the Steinhaus property.

0.4 New result. Here we develop the harmonic analysis method even more to obtain that any set S for which

$$\int_S |x|^{\frac{10}{3}+\epsilon}\, dx < \infty, \quad \text{for some } \epsilon > 0, \qquad (5)$$

cannot have the Steinhaus property ($|x|$ denotes the Euclidean norm of the vector $x \in \mathbf{R}^2$).

0.5 The problem in the Fourier domain. Notice that any measurable Steinhaus set must have measure 1. The harmonic analysis approach is based on the following simple lemma. (The definition of the Fourier transform used in this paper is

$$\widehat{f}(\xi) = \int_{\mathbf{R}^n} f(x) e^{-2\pi i \langle \xi, x \rangle}\, dx,$$

for $f \in L^1(\mathbf{R}^n)$, $n = 1, 2$.)

Lemma 1 *Let $S \subset \mathbf{R}^2$ be measurable of measure* 1. *Then S satisfies* (3) *if and only if the Fourier transform $\widehat{\mathbf{1}_S}$ of the indicator function $\mathbf{1}_S$ vanishes on all circles centered at the origin that contain a lattice point (a point with integer coordinates).*

One can easily prove this Lemma if one notices that the function which appears in (3) is a function in $L^1(\mathbf{R}^2/\mathbf{Z}^2)$ whose non-constant Fourier coefficients vanish (see [6] for the complete proof). Notice that, as a corollary of the Lemma, (3) is satisfied for a dense set Θ of orientations if and only if it is satisfied for $\Theta = [0, 2\pi)$.

The circles of Lemma 1 are exactly those with radius of the form $(m^2+n^2)^{1/2}$, where $m, n \in \mathbf{Z}$. By an old result of Landau [3] the number of those circles with radius at most R is well known to be $\sim cR^2 \log^{-1/2} R$, as $R \to \infty$, where c is a constant.

0.6 The argument in [1, 6] is as follows. Define $f : \mathbf{R} \to \mathbf{C}$ to be the restriction of $\widehat{\mathbf{1}_S}$ along the line L spanned by the unit vector $u \in \mathbf{R}^2$. The function f is then the Fourier transform of the projection of $\mathbf{1}_S$ onto L, that is of the function

$$g : \mathbf{R} \to \mathbf{R}, \quad g(t) = \int_{\mathbf{R}} \mathbf{1}_S(tu + xu^\perp)\, dx. \tag{6}$$

To make the argument clear, assume that S is bounded. Clearly then g has compact support and therefore $f = \widehat{g}$ is an entire function which satisfies (as a function of a complex argument)

$$|f(z)| \le C_1 \exp(C_2|z|), \quad \text{as } |z| \to \infty.$$

Jensen's formula [5, p. 82] then implies that $f(z)$ has at most $C_3 R$ zeros in the disk $\{z : |z| \le R\}$, as $R \to \infty$, which is clearly incompatible with the fact that f has at least as many zeros as many circles of radius $(m^2+n^2)^{1/2}$ the line L intersects, that is $\sim cR^2 \log^{-1/2} R$. The most that can be said with this approach [6] is that, for $K > 0$ sufficiently large, the area of the set in (4) cannot decay like $\exp(-KR^2 \log^{1/2} R)$ if the set S is to have the Steinhaus property.

1. Slow decay for Steinhaus sets

1.1 We shall prove the following theorem.

Theorem 1 *If $S \subset \mathbf{R}^2$ is measurable and*

$$\int_S |x|^\beta\, dx < \infty, \quad \textit{for some } \beta > \tfrac{10}{3}, \tag{7}$$

then S does not have the Steinhaus property (3).

Notation. Let a_k, $k = 1, 2, \ldots$, be the sequence of the positive numbers of the type $(m^2+n^2)^{1/2}$, $m, n \in \mathbf{Z}$, in increasing order, so that $a_1 = 1, a_2 = \sqrt{2}, a_3 = 2$, etc. Let also $\delta_k = a_{k+1} - a_k$, $k \ge 1$, be the sequence of the gaps between the a_k's. The letter C will denote an arbitrary positive constant, not necessarily the same in all its occurences. For $f : \mathbf{R} \to \mathbf{C}$, $0 \le \alpha \le 1$, define

$$\|f\|_{C^\alpha} = \sup_{x \neq y} \frac{|f(y) - f(x)|}{|y-x|^\alpha}.$$

1.2 Number-theoretic facts. According to the theorem of Landau mentioned in §0.5 we have

$$a_k \sim C k^{1/2} \log^{1/4} k. \tag{8}$$

And it is a rather elementary fact (but still the best of its kind that we know) that

$$\delta_k \le C a_k^{-1/2} \le C k^{-1/4} \log^{-1/8} k, \quad \text{as } k \to \infty. \tag{9}$$

We shall also use the following result of Hooley [4]:

For any $\gamma \in [0, \frac{5}{3})$ we have

$$\sum_{a_{k+1}^2 \le x} (a_{k+1}^2 - a_k^2)^\gamma \le C x \log^{(\gamma-1)/2} x. \tag{10}$$

Hooley's result implies

$$\sum_{2^n \le a_{k+1}^2 < 2^{n+1}} \delta_k^\gamma \le C 2^{(1-\gamma/2)n} n^{\gamma/4 - 1/2}, \tag{11}$$

for any $\gamma \in (0, \frac{5}{3})$.

1.3 Hölder functions. We shall need the following lemma.

Lemma 2 *For $f \in L^1(\mathbf{R})$, $0 \le \alpha \le 1$, we have*

$$\left\|\widehat{f}\right\|_{C^\alpha} \le C \int_{\mathbf{R}} |\xi|^\alpha |f(\xi)| \, d\xi. \tag{12}$$

Proof. Write

$$A = \int_{\mathbf{R}} |\xi|^\alpha |f(\xi)| \, d\xi.$$

We have for $h > 0$

$$\begin{aligned} \left|\widehat{f}(x+h) - \widehat{f}(x)\right| &= \left|\int_{\mathbf{R}} f(\xi)\, e^{-2\pi i x\xi}(e^{-2\pi i h\xi} - 1)\, d\xi\right| \\ &\le \int_{\mathbf{R}} |f(\xi)| \left|e^{-2\pi i h\xi} - 1\right| d\xi \\ &= \int_{|\xi|<1/h} + \int_{|\xi|>1/h} \\ &= I_1 + I_2. \end{aligned}$$

Notice that $\left|e^{-2\pi i h\xi}-1\right| \leq 2\pi h|\xi|$. Then

$$I_1 \leq 2\pi \int_{|\xi|<1/h} |f(\xi)|\, h|\xi|\ d\xi \leq 2\pi\, h \frac{1}{h^{1-\alpha}} \int_{|\xi|<1/h} |f(\xi)|\, |\xi|^\alpha\ d\xi \leq 2\pi A h^\alpha,$$

and

$$I_2 \leq 2 \int_{|\xi|>1/h} |f(\xi)|\, \frac{|\xi|^\alpha}{h^{-\alpha}}\ d\xi \leq 2Ah^\alpha,$$

which completes the proof of the Lemma. □

1.4 For a measurable set $S \subset \mathbf{R}^2$ with the Steinhaus property define the function f to be the restriction of $\widehat{\mathbf{1}_S}$ on a line L spanned by the unit vector $u \in \mathbf{R}^2$

$$f : \mathbf{R} \to \mathbf{C}, \quad f(t) = \widehat{\mathbf{1}_S}(tu),$$

and notice that f vanishes at the points $\pm a_k$, $k \geq 1$ (Lemma 1). We first prove an inequality for such functions on the line.

Lemma 3 *Let $f : \mathbf{R} \to \mathbf{C}$ vanish at the points $\pm a_k$, $k \geq 1$, and assume that $f \in C^3(\mathbf{R})$. We have then, for $\frac{1}{3} < \alpha \leq 1$,*

$$\int_{\mathbf{R}} |xf(x)|\ dx \leq C_\alpha \left\| f^{(3)} \right\|_{C^\alpha}. \tag{13}$$

It is the rapid decay of the gaps δ_k that makes (13) possible.

Proof. It is clearly sufficient to prove (13) for real valued f. Furthermore, it suffices to prove the inequality for the half line

$$\int_0^\infty |xf(x)|\ dx \leq C_\alpha \left\| f^{(3)} \right\|_{C^\alpha}. \tag{14}$$

Write $a_k^{(0)} = a_k$, for each k. By Rolle's theorem in each interval of the type $(a_k^{(0)}, a_{k+1}^{(0)}), k \geq 1$, there is a point, call it $a_k^{(1)}$, at which f' vanishes. Similarly in every interval of the type $(a_k^{(1)}, a_{k+1}^{(1)}), k \geq 1$, there is a point $a_k^{(2)}$ at which $f^{(2)}$ vanishes, etc. We so define the points $a_k^{(i)}, k \geq 1$, at which the i-th derivative of f vanishes, for $i \leq 3$.

By our construction we have $a_k^{(i+1)} - a_k^{(i)} \leq a_{k+1}^{(i)} - a_k^{(i)}$. This implies

$$0 \leq a_k^{(i)} - a_k \leq \sum_{j=0}^{C(i)} \delta_{k+j},$$

for $i \leq 3$, where $C(i)$ is a function of i only. Define

$$d_k = \left| a_{k+C(i)} - a_k \right|.$$

It follows that $d_k \leq C(i)\max\{\delta_k,\ldots,\delta_{k+C(i)}\}$ and, since we only consider $i \leq 3$, we conclude that d_k behaves essentially like δ_k, that is we have

$$d_k \leq Ck^{-1/4}\log^{-1/8} k, \tag{15}$$

and, the corrsponding inequality to (11),

$$\sum_{2^n \leq a_{k+1}^2 < 2^{n+1}} d_k^\gamma \leq C 2^{(1-\gamma/2)n} n^{\gamma/4-1/2}, \tag{16}$$

for any $\gamma \in [0, \frac{5}{3})$.

We have for $i \leq 3$, $k \geq 1$,

$$\begin{aligned}
\int_{a_k}^{a_{k+1}} |xf(x)|\,dx &\leq a_{k+1}\int_{a_k}^{a_{k+1}} |f(x)|\,dx \\
&\leq a_{k+1}\int_{a_k}^{a_{k+1}}\int_{a_k^{(0)}}^{x} |f'(y)|\,dy\,dx \\
&\leq a_{k+1}\int_{a_k}^{a_{k+1}}\int_{a_k^{(0)}}^{x}\cdots\int_{a_k^{(i-1)}}^{z} \left|f^{(i)}(w)\right|\,dw\,dz\cdots dy\,dx \\
&\leq a_{k+1}\,d_k^{i+1}\,d_k^{\alpha}\,\left\|f^{(i)}\right\|_{C^\alpha},
\end{aligned}$$

since all the regions of integration are subsets of $[a_k^{(i)}, a_{k+1}^{(i)}] \subseteq [a_k, a_k + d_k]$ and $\left|f^{(i)}(w)\right| \leq d_k^\alpha \left\|f^{(i)}\right\|_{C^\alpha}$.

Adding up for $k = 1, 2, \ldots$, we get

$$\int_{a_1}^{\infty} |xf(x)|\,dx \leq \left(\sum_{k=1}^{\infty} a_{k+1} d_k^{i+1+\alpha}\right)\left\|f^{(i)}\right\|_{C^\alpha}. \tag{17}$$

But the series in the right hand side of (17) is finite for $i = 3$, $\alpha > \frac{1}{3}$. Indeed

$$\begin{aligned}
\sum_{2^n \leq a_{k+1}^2 < 2^{n+1}} a_{k+1} d_k^{i+1+\alpha} &\leq C\,2^{n/2} n^{1/4} \sum_{2^n \leq a_{k+1}^2 < 2^{n+1}} d_k^{i+1+\alpha-\gamma} d_k^\gamma, \quad \text{by (8)}, \\
&\leq C\,2^{n/2} n^{1/4} 2^{-n\frac{i+1+\alpha-\gamma}{4}} n^{-\frac{i+1+\alpha-\gamma}{8}} \times \\
&\qquad \times \sum_{2^n \leq a_{k+1}^2 < 2^{n+1}} d_k^\gamma, \quad \text{by (15)}, \\
&\leq C\,n^{\frac{1-i-\alpha+\gamma}{8}} 2^{n\frac{1-i-\alpha+\gamma}{4}} 2^{n(1-\frac{\gamma}{2})} n^{\gamma/4-1/2}, \quad \text{by (16)}, \\
&\leq C\,n^{\frac{3\gamma-i-3-\alpha}{8}} 2^{n\frac{1}{4}(5-i-\alpha-\gamma)},
\end{aligned}$$

and for $i + \alpha > \frac{10}{3}$, we can always find $\gamma \in [0, \frac{5}{3})$ that makes $5 - i - \alpha - \gamma < 0$, which guarantees the convergence of the series in the right hand side of (17).

The inequality $\int_0^1 |xf(x)|\ dx \leq C_\alpha \left\| f^{(3)} \right\|_{C^\alpha}$ can be proved likewise. This concludes the proof of (14) and the Lemma.□

1.5 Proof of Theorem 1. Assume that $S \subset \mathbf{R}^2$ has the Steinhaus property and satisfies $\int_S |x|^\beta\ dx < \infty$, for some $\beta > \frac{10}{3}$. Write $u_\theta = (\cos\theta, \sin\theta)$ and also write $\Pi_\theta \mathbf{1}_S$ for the projection of $\mathbf{1}_S$ on the line spanned by u_θ (see (6) for the definition). Notice that for all θ

$$\int_{\mathbf{R}} |x|^\beta (\Pi_\theta \mathbf{1}_S)(x)\ dx = \int_S |\langle x, u_\theta \rangle|^\beta\ dx \leq \int_S |x|^\beta\ dx < \infty.$$

An immediate consequence of this is that the function $\mathbf{R} \to \mathbf{R}$, $x \to \widehat{\mathbf{1}_S}(xu_\theta)$, is three times continuously differentiable, for every θ, being the one-dimensional Fourier transform of $\Pi_\theta \mathbf{1}_S$, and the third derivative has finite $C^{\beta-3}$ norm (notice $\beta - 3 > \frac{1}{3}$). This function also vanishes at the points $\pm a_k$ (the radii of the circles centered at the origin that go through a lattice point) according to Lemma 1.

The only way in which we use the fact that $\mathbf{1}_S$ is an indicator function is to observe that its Fourier transform cannot be absolutely integrable, $\mathbf{1}_S$ being discontinuous. We have thus

$$\begin{aligned} \infty &= \int_{\mathbf{R}^2} \left| \widehat{\mathbf{1}_S}(x) \right|\ dx \\ &= \int_0^\pi \int_{\mathbf{R}} \left| x \widehat{\mathbf{1}_S}(xu_\theta) \right|\ dx\ d\theta \\ &\leq C_\beta \int_0^\pi \left\| \frac{d^3}{dx^3} \widehat{\mathbf{1}_S}(xu_\theta) \right\|_{C^{\beta-3}}\ d\theta, \quad \text{by Lemma 3.} \end{aligned}$$

But, by Lemma 2, for any function $\varphi \in L^1(\mathbf{R})$ we have $\left\| \frac{d^3}{dx^3} \widehat{\varphi}(x) \right\|_{C^{\beta-3}} \leq C \left\| x^\beta \varphi(x) \right\|_1$. Therefore, remembering that $x \to \widehat{\mathbf{1}_S}(xu_\theta)$ is the Fourier transform of $\Pi_\theta \mathbf{1}_S$, we get

$$\begin{aligned} \infty &= \int_0^\pi \int_{\mathbf{R}} |x|^\beta (\Pi_\theta \mathbf{1}_S)(x)\ dx\ d\theta \\ &= \int_0^\pi \int_S |\langle x, u_\theta \rangle|^\beta\ dx\ d\theta \\ &\leq \pi \int_S |x|^\beta\ dx, \end{aligned}$$

a contradiction, and the proof is complete.□

1.6 Remark. Even without using Hooley's result (10) one can still prove that any Steinhaus set S must necessarily have $\int_S |x|^{4+\epsilon}\ dx = \infty$, for any $\epsilon > 0$. Just using inequality (9) one can prove, in place of Lemma 3, the inequality $\int_{\mathbf{R}} |xf(x)|\ dx \leq C_\epsilon \left\| f^{(4)} \right\|_{C^\epsilon}$, for any $\epsilon > 0$ (this corresponds to $\gamma = 1$ in (10)).

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