The study of translational tiling with Fourier Analysis

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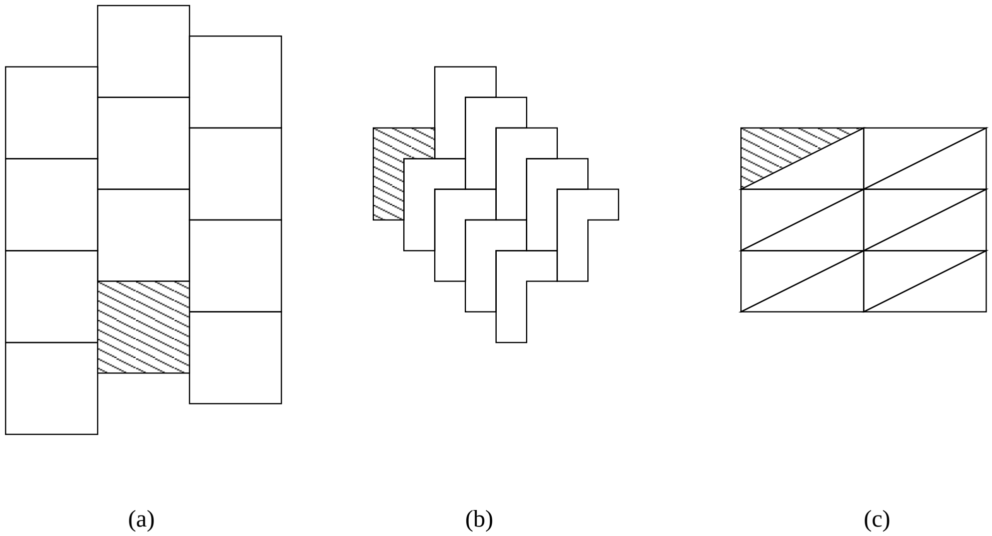

FIGURE 1: Examples of tiling with the shaded objects. In (c) a tiling by a triangle is shown that is using rotations as well as translations. We will not deal with such tilings here. In (a) a tiling by a square is shown and in (b) a tiling by an L-shaped region. In (b) the set of translations is a lattice, but not in (a).

Forward

In this survey I will try to describe how Fourier Analysis is used in the study of translational tiling. Right away I will emphasize two restrictions that separate this area from the general theory of tilings.

- There is only one tile. This is an object that is moved around in space (whatever space we are trying to tile, most generally an abelian group) in a way that there are no "overlaps" among the several copies of it and almost nothing, in the sense of Lebesgue or counting measure, is left uncovered. This object may be a domain in space or a function defined on space, usually nonnegative. Examples are shown in Figure 1.
- The only allowed motions of the tile are translations. No rotations or reflections of the object are allowed. In fancier language, we are tiling abelian groups, not vector spaces.

This paper is broken up into three "lectures", which correspond roughly to the three hour-long lectures I gave in the Universitá di Milano–Bicocca, in June 2001, during the meeting on *Fourier Analysis and Convexity*. Lecture 1 has to do with how Fourier Analysis is used to prove structure, or rigidity, in tilings. In Lecture 2, some problems are presented about lattice-tiling and in Lecture 3 a tiling problem of Functional Analysis is discussed, the Fuglede Conjecture on spectral domains.

An advance apology: I will describe mostly material with which I am aquainted the most, through my own work.

Finally, I would like to thank the organizers L. Brandolini, L. Colzani, A. Iosevich and G. Travaglini for organizing this great meeting and giving me the chance to participate.

1 Lecture 1: Introduction to the method and structure of tilings.

1.1 Tiling and density

It's time for the first definition, of what tiling means. We speak mostly of tiling $\mathbb{R}^d$ and $\mathbb{Z}^d$ in this paper, but tiling makes sense on all abelian groups.

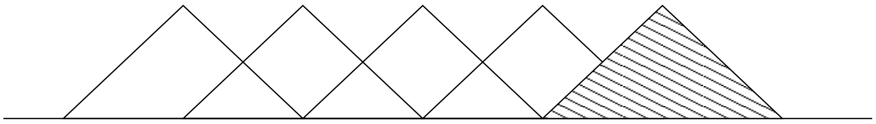

FIGURE 2: A triangle function tiling the real line

Definition 1.1. (Translational tiling)
Suppose $0 \le f \in L^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ is a discrete multiset. We say that f tiles $\mathbb{R}^d$ with Λ at level (or weight) ℓ if

$$\sum_{\lambda \in \Lambda} f(x-\lambda) = \ell, \quad a.e.(x).$$

We write: $f + \Lambda = \ell\, \mathbb{R}^d$.

In Figure 2 a tiling by the triangle function $f(x) = (1-|x|)^+$ is shown with translation set $\Lambda = \mathbb{Z}$ and level 1. In the particular case when $f = \chi_\Omega$ is the indicator function of a measurable domain $\Omega \subseteq \mathbb{R}^d$ of finite measure, we write also $\Omega + \Lambda = m\mathbb{R}^d$, where the positive integer m represents the level of the (generally multiple) tiling.

The tiling assumption $f + \Lambda = \ell\mathbb{R}^d$ has some immediate implications about the density properties of the multiset Λ.

Definition 1.2. (Density)
A multiset $\Lambda \subseteq \mathbb{R}^d$ has asymptotic density ρ if

$$\lim_{R\to\infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|} \to \rho$$

uniformly in $x \in \mathbb{R}^d$. We write $\rho = \mathrm{dens}\,\Lambda$.

We say that Λ has (uniformly) bounded density if the fraction above is bounded by a constant ρ uniformly for $x \in \mathbb{R}$ and $R > 1$. We say then that Λ has density (uniformly) bounded by ρ.

Last, the upper density of a set $\Lambda \subseteq \mathbb{R}^d$ is defined as

$$\limsup_{R\to\infty} \sup_{x\in\mathbb{R}^d} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|}.$$

Remark 1.1. *According to this definition a set Λ may have* density uniformly bounded by a number $\rho < \infty$ *yet* $\mathrm{dens}\,\Lambda$ *may not exist.*

Lemma 1.1. *If $0 \le f \in L^1(\mathbb{R}^d)$ is not the zero function and $f + \Lambda = \ell\mathbb{R}^d$ then Λ has bounded density.*

Proof. By hypothesis

$$\sum_{a\in\Lambda} f(x-a) = \ell, \quad \text{almost everywhere,}$$

and clearly $\ell > 0$. Choose $R > 1$ so that $J = \int_{B_R(0)} f > 0$, where $B_R(0)$ is the ball centered at 0 with radius R. Let $t \in \mathbb{R}^d$ be arbitrary. We have

$$\begin{aligned} |B_{2R}(0)| \cdot \ell &= \int_{B_{2R}(t)} \sum_{a\in\Lambda} f(x-a)\ dx \\ &\geq \int_{B_{2R}(t)} \sum_{|a-t|<R} f(x-a)\ dx \\ &\geq \#(\Lambda \cap B_R(t)) \int_{B_R(0)} f. \end{aligned}$$

Thus $\#(\Lambda \cap B_R(t)) \leq |B_{2R}(0)|\ell/J$ is bounded independent of t, which implies that Λ has uniformly bounded density. ∎

Working similarly on easily gets the following lemma.

Lemma 1.2. *If $0 \leq f \in L^1(\mathbb{R}^d)$ is not the zero function and $f + \Lambda = \ell\mathbb{R}^d$ then Λ has density* $\operatorname{dens}\Lambda = \ell(\int f)^{-1}$.

It is time also to define *packing.*

Definition 1.3. (Packing)
Suppose $0 \leq f \in L^1(\mathbb{R}^d)$ and $\Lambda \subseteq \mathbb{R}^d$ is a discrete multiset. We say that f packs $\mathbb{R}^d$ with Λ at level ℓ if

$$\sum_{\lambda\in\Lambda} f(x-\lambda) \leq \ell, \quad a.e.(x).$$

We write: $f + \Lambda \leq \ell\,\mathbb{R}^d$.

The following lemma is almost trivial, yet useful.

Lemma 1.3. *If $0 \leq f \in L^1(\mathbb{R}^d)$ is not the zero function and $f + \Lambda \leq \ell\mathbb{R}^d$ is a packing then Λ has density uniformly bounded by $\ell(\int f)^{-1}$.*

Finally, one can easily prove the following about translation sets.

Lemma 1.4. *Suppose $f + \Lambda \leq \ell\mathbb{R}^d$ and* $\operatorname{esssup} f = \ell$. *Then*

$$\inf\{|\lambda - \mu| :\ \lambda, \mu \in \Lambda, \lambda \neq \mu\} > 0. \tag{1.1}$$

In particular, if $E + \Lambda = \mathbb{R}^d$ is a tiling by the set E at level 1 then (1.1) *holds.*

1.2 Tiling in Fourier space

Next, we associate to any point multiset Λ the measure

$$\delta_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda,$$

where δ_λ is one unit point mass at the point λ (see Figure 3). Generally, this measure is infinite globally but has finite total variation in any bounded set, at least when the set Λ has bounded density. This is the case whenever Λ is involved in a tiling. It follows that

$$|\delta_\Lambda|(B_R(t)) \leq CR^d,$$

which implies that the object δ_Λ is a so-called tempered distribution, a bounded linear functional on the Schwarz space $\mathcal{S}$ of smooth functions which, along with all their partial derivatives, decay faster than any power at infinity.

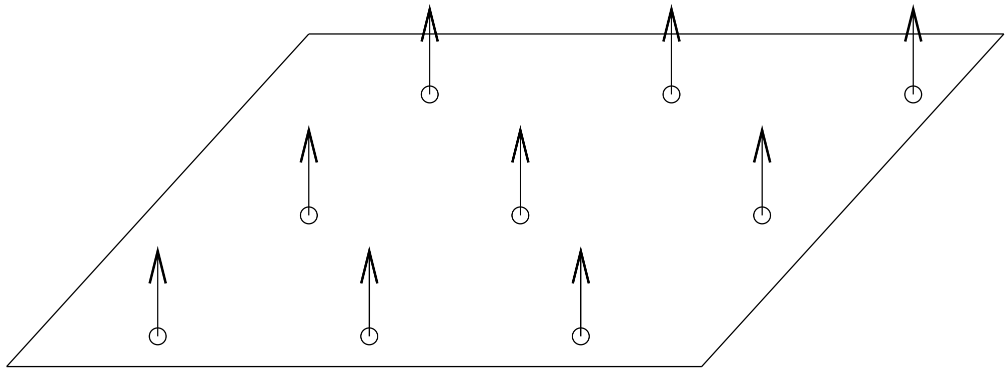

FIGURE 3: The measure δ_Λ corresponding to some Λ in the plane.

If T is a tempered distribution one defines its Fourier Transform $\widehat{T}$ by duality as follows:

$$\widehat{T}(\phi) = T(\widehat{\phi}),$$

for any $\phi \in \mathcal{S}$ (it is easy to prove that the Fourier Transform $\widehat{\phi}$ is also in $\mathcal{S}$). We normalize the Fourier Transform for a function $f \in L^1(\mathbb{R}^d)$ as

$$\widehat{f}(t) = \int e^{-2\pi i \langle t, x\rangle} f(x)\ dx,$$

which leads to the inversion formula

$$f(x) = \int e^{2\pi i \langle t, x\rangle} \widehat{f}(t)\ dt,$$

whenever $\widehat{f} \in L^1$, which happens for all functions $f \in \mathcal{S}$.

We are now in the position to argue formally as follows. Suppose $f + \Lambda = \ell\mathbb{R}^d$. This means that

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = \ell, \quad \text{(a.e. } x),$$

which we rewrite as a convolution

$$f * \delta_\Lambda = \ell.$$

Take the Fourier Transform of both sides to get

$$\widehat{f} \cdot \widehat{\delta_\Lambda} = \ell\delta_0.$$

As the support of the right hand side is just $\{0\}$ we conclude that

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup \mathcal{Z}(\widehat{f}), \tag{1.2}$$

where we denote the zero-set of the continuous function g by $\mathcal{Z}(g)$:

$$\mathcal{Z}(g) = \left\{x \in \mathbb{R}^d :\ g(x) = 0\right\}.$$

The inclusion in (1.2) is the starting point of the method of applying Fourier Analysis to translational tiling. Whenever we have tiling, we deduce (1.2). Sometimes we may be able to get tiling from (1.2), but we usually need some extra conditions to make this conclusion.

Having argued formally, let us now prove carefully the following theorem. Notice that we have essentially added the condition $\widehat{f} \in C^\infty$ to make the argument go through. This condition is automatically valid whenever f has compact support, as, for instance, when f is the indicator function of a bounded domain (the classical geometric situation), but will definitely not be there when we talk about the Fuglede problem in Lecture 3. There we will need a different theorem of this sort, with different assumptions (see Theorem 3.11).

Theorem 1.1. *Suppose that $f \in L^1(\mathbb{R}^d)$ is nonnegative, $\widehat{f} \in C^\infty$ and $f + \Lambda = \ell\mathbb{R}^d$ for some multiset Λ. Then (1.2) follows.*

Proof. Let $K = \{0\} \cup \mathcal{Z}(\widehat{f})$, which is a closed set. Inclusion (1.2) means (by the definition of the support of a tempered distribution) that $\widehat{\delta_\Lambda}(\psi) = 0$ for all smooth ψ supported in K^c (see Figure 4). For such a ψ

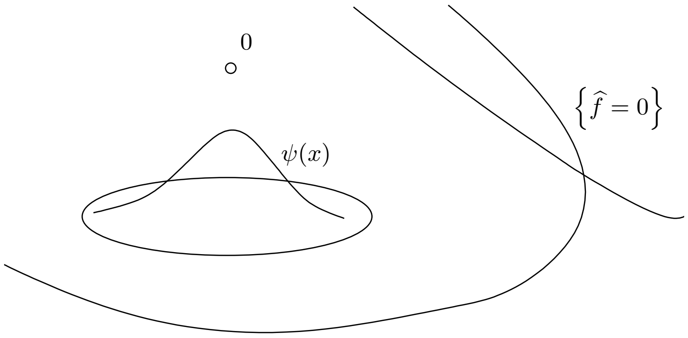

FIGURE 4: A test function ψ supported away from $\{0\} \cup \left\{\widehat{f} = 0\right\}$

$$
\begin{aligned}
\left(\widetilde{\widehat{f}}\psi\right)^\wedge(\lambda) &= \int \overline{\widehat{f}(-x)}\psi(x)e^{-2\pi i\lambda x}\,dx \\
&= \int\int \overline{f(y)}e^{-2\pi iyx}\psi(x)e^{-2\pi i\lambda x}\,dx\,dy \\
&= \int \overline{f(y)} \int \psi(x)e^{-2\pi i(\lambda+y)x}\,dx\,dy \\
&= \int \overline{f(y)}\widehat{\psi}(\lambda+y)\,dy \\
&= \int \overline{f(-t)}\widehat{\psi}(\lambda-t)\,dt \\
&= (\widetilde{f} * \widehat{\psi})(\lambda), \qquad (1.3)
\end{aligned}
$$

where we use the notation $\widetilde{f}(x) = \overline{f(-x)}$.

We must show $\widehat{\delta_\Lambda}(\psi) = 0$. We have

$$\widehat{\delta_\Lambda}(\psi) = \widehat{\delta_\Lambda}\left(\widetilde{\widehat{f}} \cdot \frac{\psi}{\widetilde{\widehat{f}}}\right).$$

Notice that $\widetilde{\widehat{f}}$ and $\widehat{f}$ have the same zeros (since f is real), so the quotient $\phi = \psi/\widetilde{\widehat{f}}$ is a $C_0^\infty(K^c)$ function.

We have

$$\begin{aligned}
\widehat{\delta_\Lambda}(\psi) &= \widehat{\delta_\Lambda}(\widetilde{\widetilde{f}}\,\phi) \\
&= \delta_\Lambda\left((\widetilde{\widetilde{f}}\phi)^\wedge\right) \quad \text{(by the definition of the Fourier Transform for distributions)} \\
&= \sum_{\lambda\in\Lambda} (\widetilde{\widetilde{f}}\phi)^\wedge(\lambda) \quad \text{(by the definition of } \delta_\Lambda) \\
&= \sum_{\lambda} (\widetilde{f} * \widehat{\phi})(\lambda) \quad \text{(by (1.3))} \\
&= \sum_{\lambda} \int \widetilde{f}(\lambda - x)\widehat{\phi}(x)\,dx \\
&= \int \sum_{\lambda} \overline{f(x-\lambda)}\widehat{\phi}(x)\,dx \\
&= \ell \int \widehat{\phi}(x)\,dx \quad (\text{since } f + \Lambda = \ell\mathbb{R}^d) \\
&= \ell\phi(0) \\
&= 0 \quad (\text{as } 0 \notin \operatorname{supp}\phi).
\end{aligned}$$

∎

1.2.1 The lattice case and the sufficiency of the support condition for tiling

Suppose $\Lambda = A\mathbb{Z}^d$, $A \in GL(d, \mathbb{R})$, is a lattice in $\mathbb{R}^d$ (a discrete subgroup which contains d linearly independent vectors). The Fourier Transform of the tempered distribution δ_Λ takes a particularly simple form as claimed by the *Poisson Summation Formula:*

$$\widehat{\delta_\Lambda} = \frac{1}{\det A}\delta_{\Lambda^*}, \tag{1.4}$$

where

$$\Lambda^* = \left\{\xi \in \mathbb{R}^d : \ \langle \xi, \lambda\rangle \in \mathbb{Z}, \quad \forall \lambda \in \Lambda\right\} = A^{-\top}\mathbb{Z}^d$$

is the *dual lattice* of Λ (see Figure 5).

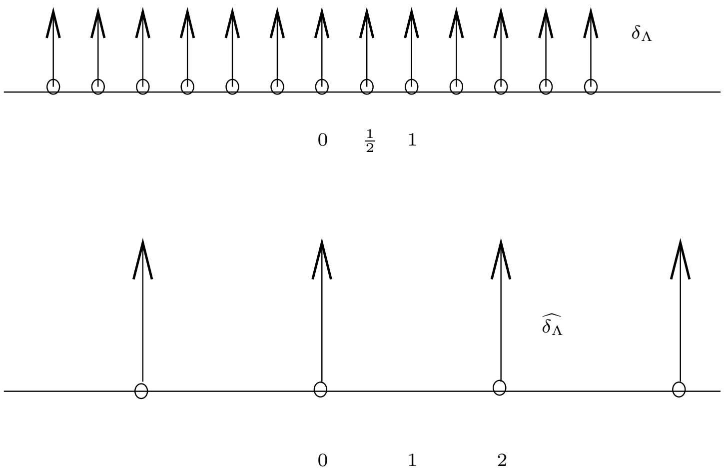

FIGURE 5: The "Dirac comb" δ_Λ when $\Lambda = \frac{1}{2}\mathbb{Z}$, and its Fourier Transform, the comb $2\delta_{2\mathbb{Z}}$.

The Poisson Summation Formula is usually stated as the equality

$$\sum_{\lambda\in\Lambda} \widehat{\phi}(\lambda) = \frac{1}{\det A} \sum_{\lambda^*\in\Lambda^*} \phi(\lambda^*),$$

for all $\phi \in \mathcal{S}$, and this is exactly the content of (1.4), as the Fourier Transform of δ_Λ is defined by duality.

Equation (1.2) now gives the implication below, valid for any lattice Λ,

$$f + \Lambda \text{ is a tiling} \Rightarrow \widehat{f} \text{ vanishes on } \Lambda^* \setminus \{0\}.$$

This is in fact easy to prove using ordinary multiple Fourier Series, after applying a linear transformation that maps Λ to $\mathbb{Z}^d$. Working this way one gets easily that the above implication is, in fact, an equivalence, so that

$$f + \Lambda \text{ is a tiling} \Leftrightarrow \widehat{f} \text{ vanishes on } \Lambda^* \setminus \{0\}. \tag{1.5}$$

We prefer however to stick to using (1.2) as our guiding tool and not mention Fourier Series. As to why the reverse implication holds, the answer is in the following theorem.

Theorem 1.2. *Suppose that Λ is a multiset of bounded density and that f is a nonnegative integrable function on $\mathbb{R}^d$. Suppose also that $\widehat{\delta_\Lambda}$ is locally a measure and that*

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup \left\{ \widehat{f} = 0 \right\}.$$

Then Λ has density and $f + \Lambda = \ell\mathbb{R}^d$, for $\ell = \int f \cdot \operatorname{dens} \Lambda$.

Intuitively, to kill a tempered distribution which is a measure any zero (of whatever order) suffices.

Proof. Let $F(x) = \sum_{\lambda\in\Lambda} f(x-\lambda)$. We want to show that F is a constant ℓ and for this it is enough to show that for any nonnegative $\widehat{\phi} \in \mathcal{S}$ we have $\int F\widehat{\phi} = \ell \int \widehat{\phi} = \ell\phi(0)$. We have

$$\begin{aligned}
\int F\widehat{\phi} &= \sum_\lambda \int f(x-\lambda)\widehat{\phi}(x)\, dx \\
&= \int f(y) \sum_\lambda \widehat{\phi}(y+\lambda)\, dy \\
&= \int f(y)\delta_\Lambda(\widehat{\phi}(y-\cdot))\, dy \\
&= \int f(y)\widehat{\delta_\Lambda}(e^{2\pi i yx}\phi(x))\, dy \\
&= \int\int f(y)e^{2\pi i yx}\phi(x)\, d\widehat{\delta_\Lambda}(x)\, dy \\
&= \int \widehat{f}(-x)\phi(x)\, d\widehat{\delta_\Lambda}(x) \\
&= \widehat{\delta_\Lambda}(\{0\})\phi(0)\widehat{f}(0),
\end{aligned}$$

which proves the desired equality with $\ell = \int f \cdot \widehat{\delta_\Lambda}(\{0\})$. The fact that Λ has density and the value for $\operatorname{dens}\Lambda$ follow from Lemma 1.2. ∎

1.3 Structure of tilings in dimension 1

We can now show the following theorem [**LM91**, **KL96**].

Theorem 1.3. (Leptin and Müller, 1991, and Kolountzakis and Lagarias, 1996) *Suppose $0 \le f \in L^1(\mathbb{R})$ and has compact support. Suppose also that*

$$f + \Lambda = \ell\mathbb{R},$$

for some $\Lambda \in \mathbb{R}$. Then there are $J \in \mathbb{N}$, $\alpha_j, \beta_j \in \mathbb{R}$, $j = 1, \ldots, J$, $\alpha_j > 0$, such that

$$\Lambda = \bigcup_{j=1}^{J} (\alpha_j \mathbb{Z} + \beta_j).$$

That is, tiling sets for compactly supported tiles in dimension 1 are finite unions of complete arithmetic progressions.

Remark 1.2. *Only recently (after the first draft of this survey was written in March 2003) has the author learned of the work of Leptin and Müller [**LM91**]. In [**LM91**] Theorem 1.3 was proved, by essentially the same method.*

1.3.1 The idempotent theorem, the Bohr group and Meyer's theorem

This extreme structure is, in the end, a consequence of P.J. Cohen's idempotent theorem on a general abelian group [**Coh59**].

Theorem 1.4. (Cohen, 1959)
If $\mu \in M(G)$ is a finite measure on a locally compact abelian group G, such that $\widehat{\mu}$ takes only finitely many values then, for any such value c, the set $S = \left\{\gamma \in \widehat{G} : \ \widehat{\mu}(\gamma) = c\right\}$ belongs to the open coset ring of $\widehat{G}$.

The (open) coset ring is defined below.

Definition 1.4. (The coset ring of a group)
The coset ring *of an abelian group G is the smallest collection of subsets of G which is closed under finite unions, finite intersections and complements and which contains all cosets of G. For a topological group G the smallest ring of subsets of G which contains all* open *cosets is called the* open coset ring *of G.*

Cohen's theorem therefore says that S can be constructed with finitely many set-theoretic operations from the open cosets of $\widehat{G}$.

The group $\widehat{G}$ is called the *dual group* of G and is the group of continuous characters on G, that is, the group of all group homomorphisms $G \to \mathbb{C}$ with the group operation beeing the pointwise multiplication. It can be proved that $\widehat{\widehat{G}}$ is isomorphic (as a topological group) with G (Pontryagin duality) and that $\widehat{G}$ is compact if and only if G is discrete. Further $\widehat{G \times H} = \widehat{G} \times \widehat{H}$. Some dual group pairs are the following: $(\mathbb{Z}, \mathbb{T})$, $(\mathbb{R}, \mathbb{R})$, $(\mathbb{Z}_n, \mathbb{Z}_n)$, $(\mathbb{R}^d, \mathbb{R}^d)$, $(\mathbb{Z}^d, \mathbb{T}^d)$.

If μ is a finite measure on G its Fourier Transform is a continuous function on $\widehat{G}$ defined by

$$\widehat{\mu}(\xi) = \int_G \overline{\xi(x)} \, d\mu(x),$$

the integration carried out with respect to the essentially unique translation invariant measure on G called the *Haar measure*. For example, when $G = \mathbb{R}$ the Haar measure is Lebesgue measure and $\xi(x) = e^{2\pi I \xi x}$. (The reader should consult [**R62**] for the basic definitions and facts about Fourier Analysis on locally compact abelian groups.)

We do not use Cohen's theorem directly, but rather a consequence of it discovered by Y. Meyer [**Mey70**].

Theorem 1.5. (Meyer, 1970)
Let $\Lambda \subseteq \mathbb{R}^d$ be a discrete set and δ_Λ be the Radon measure

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} c_\lambda \delta_\lambda, \quad c_\lambda \in S,$$

where $S \subseteq \mathbb{C} \setminus \{0\}$ is a finite set. Suppose that δ_Λ is tempered, and that $\widehat{\delta_\Lambda}$ is a Radon measure on $\mathbb{R}^d$ which satisfies

$$\left|\widehat{\delta_\Lambda}\right|(B_R(0)) \le C_1 R^d, \ \text{ as } R \to \infty, \tag{1.6}$$

where $C_1 > 0$ is a constant. Then, for each $s \in S$, the set

$$\Lambda_s = \{\lambda \in \Lambda : \ c_\lambda = s\}$$

is in the coset ring of $\mathbb{R}^d$.

Proof. Let $\phi \in C_c^\infty(B_1(0))$, $\phi(0) = 1$, so that its Fourier Transform satisfies $\left|\widehat{\phi}(\xi)\right| \le C_\alpha |\xi|^{-\alpha}$ for all $\alpha > 0$. Let μ be the measure $\sum_{\lambda\in\Lambda} \delta_\lambda$ and for positive integers n define the *functions*

$$\mu_n(x) = \phi(nx) * \mu(x).$$

Their Fourier Transforms satisfy

$$\widehat{\mu}_n(\xi) = \frac{1}{n^d}\widehat{\phi}(\xi/n)\widehat{\mu}(\xi),$$

hence the $\widehat{\mu}_n$ are all measures. We claim that the measures $\widehat{\mu_n}$ are uniformly bounded measures, i.e. $|\widehat{\mu_n}|(\mathbb{R}^d) \le C$, where C is independent of n. Indeed

$$|\widehat{\mu_n}|(B_n(0)) \le \frac{1}{n^d}||\widehat{\phi}||_\infty \, |\widehat{\mu}|(B_n(0)) \le C_1||\widehat{\phi}||_\infty, \tag{1.7}$$

by our assumption on the growth of $|\widehat{\mu}|(B_n(0))$.

Furthermore, if $2^k \gg n$ we have (using the fact that $\left|\widehat{\phi}(\xi)\right| \le C|\xi|^{-d-1}$ as $\xi \to \infty$)

$$\begin{aligned}
|\widehat{\mu_n}|(B_{2^{k+1}}(0) \setminus B_{2^k}(0)) &\le C\frac{1}{n^d}\left|\widehat{\phi}\right|_{B_{2^{k+1}/n}(0)\setminus B_{2^k/n}(0)} |\widehat{\mu}|(B_{2^{k+1}}(0)) \\
&\le C\frac{1}{n^d}\left(\frac{2^k}{n}\right)^{-d-1} 2^{(k+1)d} \\
&\le Cn2^{-k}.
\end{aligned}$$

Hence

$$|\widehat{\mu_n}|(B_n(0)^c) \le \sum_{n\le 2^k} |\widehat{\mu_n}|(B_{2^{k+1}}(0) \setminus B_{2^k}(0)) \le Cn\sum_{n\le 2^k} 2^{-k} \le C1,$$

which, together with (1.7), shows that the sequence $|\widehat{\mu_n}|(\mathbb{R}^d)$ is bounded.

Notice also that $\lim_{n\to\infty} \mu_n(x) = c_x$ if $x \in \Lambda$ and is 0 otherwise. This is a consequence of the fact that Λ is discrete and the support of $\phi(nx)$ shrinks to 0.

We now use the following properties of $\overline{\mathbb{R}^d}$, the Bohr compactification of $\mathbb{R}^d$, a locally compact abelian group.

1. $\overline{\mathbb{R}^d}$ is the dual group of $\mathbb{R}^d_\delta$, the d-dimensional Euclidean space with the discrete topology. Therefore $\overline{\mathbb{R}^d}$ is a compact group being the dual group of a discrete group.
2. $\mathbb{R}^d \subseteq \overline{\mathbb{R}^d}$ as topological spaces and $\mathbb{R}^d$ is dense in $\overline{\mathbb{R}^d}$. Identifying the continuous functions on $\overline{\mathbb{R}^d}$ with bounded continuous functions on $\mathbb{R}^d$ we get that

$$C(\overline{\mathbb{R}^d}) \subseteq C(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$$

   is a Banach space inclusion.

Since the measures $\widehat{\mu_n}$ are uniformly bounded they act on all bounded continuous functions on $\mathbb{R}^d$, and consequently also on all continuous functions on $\overline{\mathbb{R}^d}$. That is they constitute a uniformly bounded family of linear functionals on $C(\overline{\mathbb{R}^d})$. By the Banach-Alaoglu theorem there exists a measure ν on $\overline{\mathbb{R}^d}$ such that for every $f \in C(\overline{\mathbb{R}^d})$ there is a subsequence of $\widehat{\mu_n}$, call it again $\widehat{\mu_n}$, such that

$$\widehat{\mu_n}(f) \to \nu(f), \quad \text{as } n \to \infty.$$

Applying this with each character of $\overline{\mathbb{R}^d}$ in place of f we obtain that

$$\widehat{\nu}(x) = \lim_{n\to\infty} \widehat{\widehat{\mu_n}}(x) = c_{-x}, \quad \text{if } -x \in \Lambda,$$

and is 0 otherwise. Hence $\widehat{\nu}$ has the finite range $-S$. By Theorem 1.4 the set $-\Lambda$, and thus Λ, belongs to the open coset ring of $\mathbb{R}^d_\delta$. Since $\mathbb{R}^d_\delta$ has the discrete topology the open coset ring is the same as the coset ring of $\mathbb{R}^d$. ∎

Since we need to know what kind of sets the elements of the coset ring of $\mathbb{R}^d$ are, we use the following general theorem [**K00a**], which says that discrete elements of the coset ring can always be constructed from *discrete* cosets using finitely many unions, intersections and complementations.

Theorem 1.6. (Kolountzakis, 2000)
Let G be a topological abelian group and let $\mathcal{R}$ be the least ring of sets which contains the discrete cosets of G. Then $\mathcal{R}$ contains all discrete elements of the coset ring of G.

In dimension 1 this implies the following result by Rosenthal [**Ros66**].

Theorem 1.7. (Rosenthal, 1966)
The elements of the coset ring of $\mathbb{R}$ which are discrete in the usual topology of $\mathbb{R}$ are precisely the sets of the form

$$F \triangle \bigcup_{j=1}^{J} (\alpha_j \mathbb{Z} + \beta_j) \ , \tag{1.8}$$

where $F \subseteq \mathbb{R}$ is finite, $J \in \mathbb{N}$, $\alpha_j > 0$ and $\beta_j \in \mathbb{R}$ ($\triangle$ denotes symmetric difference).

1.3.2 Getting structure in dimension 1

In this section we prove Theorem 1.3. Assume that $\Lambda \subset \mathbb{R}$ is set of bounded density and that $f + \Lambda = \ell\mathbb{R}$ for a function $f \in L^1$ of compact support, contained in, say, $(-A, A)$. We will use (1.2), so the first thing to do is to obtain information on the set $\mathcal{Z}(\widehat{f}) = \left\{\widehat{f} = 0\right\}$.

We look at the Fourier Transform of f defined on the complex numbers

$$\widehat{f}(z) = \int_{\mathbb{R}} e^{-2\pi i z x} f(x)\, dx, \quad (z \in \mathbb{C}).$$

Since f is supported in $(-A, A)$ it follows that $\widehat{f}$ is entire so that $\mathcal{Z}(\widehat{f})$ is a discrete subset of $\mathbb{R}$. Furthermore $\widehat{f}$ satisfies the growth bound

$$\left|\widehat{f}(z)\right| \le \int_{-A}^{A} e^{2\pi x Im(z)} |f(x)|\, dx \le ||f||_1 e^{2\pi A|z|}.$$

If $N(T)$ counts the number of zeros of $\widehat{f}(z)$ in the disk $\{z : \ |z| \le T\}$, an application of Jensen's formula gives

$$\limsup_{T\to\infty} \frac{N(T)}{T} \le CA.$$

Write B for the discrete set $\{0\} \cup \mathcal{Z}(\widehat{f})$, so that by (1.2) the tempered distribution $\widehat{\delta_\Lambda}$ is supported on B. It is well known, and easy to prove, that a tempered distribution supported at a single point b is necessarily a finite linear combination of derivatives of δ_b, and the same proof gives that

$$\widehat{\delta_\Lambda} = \sum_{b \in B} P_b(\partial)\delta_b.$$

Here $P_b(\partial) = \sum_{j=0}^N c_j \frac{\partial^j}{\partial x^j}$ is differential polynomial operator applied on the Dirac point mass at b. (The degree N can be taken the same for all $b \in B$ as any tempered distribution has finite degree. This is not used below.)

Step 1 All P_b are constants (hence $\widehat{\delta_\Lambda}$ is locally a measure)

Focus on a single $b \in B$ and let ϕ be a smooth function of compact support. Examine the quantity

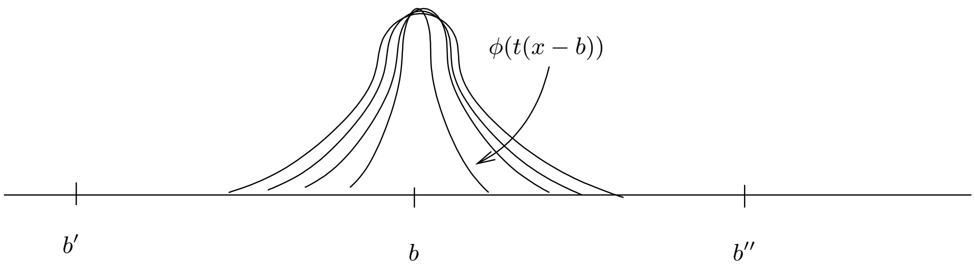

FIGURE 6: Picking out the distribution $\widehat{\delta_\Lambda}$ at b by applying it on $\phi(t(x-b))$. For large t the other points of set B are left out and the behavior at b is isolated.

$$I(t) = \widehat{\delta_\Lambda}\left(\phi(t(x-b))\right), \quad (t \to \infty),$$

as shown in Figure 6. For large t this equals

$$\begin{aligned}
\left(P_b(\partial)\delta_b\right)\left(\phi(t(x-b))\right) &= \left(\sum_{j=0}^N c_j \delta_b^{(j)}\right)\left(\phi(t(x-b))\right) \\
&= \sum_{j=1}^N c_j (-1)^j \phi^{(j)}(0) t^j.
\end{aligned}$$

Choose $\phi^{(j)}(0) = (-1)^j$ to get the above expression equal to

$$\sum_{j=1}^N c_j t^j.$$

Next we will bound the growth of $I(t)$.

Let

$$g(x) = \phi(t(x-b)), \quad \widehat{g}(\xi) = \frac{1}{t} e^{-2\pi i b\xi/t} \widehat{\phi}\left(\frac{\xi}{t}\right).$$

By duality

$$\begin{aligned}
|I(t)| &= \left|\widehat{\delta_\Lambda}(g)\right| \\
&= |\delta_\Lambda(\widehat{g})| \\
&\le \frac{1}{t}\sum_\lambda \left|\widehat{\phi}\left(\frac{\lambda}{t}\right)\right| \\
&= \frac{1}{t}\sum_{|\lambda|\le t} + \frac{1}{t}\sum_{|\lambda|>t} \\
&\le C + C\sqrt{t}\sum_{n=\lfloor t\rfloor}^{\infty} n^{-3/2} \\
&= O(\sqrt{t}).
\end{aligned}$$

We used the bounded density of Λ for the convergence of the sum $\sum_{|\lambda|>t}$, and the fact that

$$\left|\widehat{\phi}(\xi)\right| = O\left(|\xi|^{-M}\right) \tag{1.9}$$

for any $M > 0$ we wish. We took $M = 3/2$.

Since $I(t)$ cannot even grow linearly it follows that the degree N is zero and we can now write

$$\widehat{\delta_\Lambda} = \sum_{b\in B} c_b\delta_b,$$

for some constants c_b.

Step 2 The coefficients c_b are uniformly bounded.

To prove this we are just a bit more careful in the last estimate and now use a ϕ which is 1 at 0. For large t then

$$c_b = \widehat{\delta_\Lambda}(\phi(t(x-b))),$$

and one can get a bound for this by duality which does not involve t at all using the exponent $M = 2$ instead of $M = 3/2$ in (1.9).

Step 3 Use of Meyer's Theorem

Now the crucial condition

$$\left|\widehat{\delta_\Lambda}\right|(-R, R) \le CR$$

in Meyer's Theorem holds (remember there is a linear number of zeros and at each one we have a bounded mass), hence, by Rosenthal's Theorem 1.7,

$$\Lambda = \bigcup_{j=1}^{J} (\alpha_j\mathbb{Z} + \beta_j) \triangle F$$

for some real numbers α_j, β_j and finite set F.

Step 4 F is empty

Otherwise $\widehat{\delta_\Lambda}$ would have a continuous part, a trigonometric polynomial due to F. But it cannot have such a continuous part as its support is discrete.

Open Problem 1. *Is the main theorem true if f is only supposed to be in L^1 but not of compact support?*

What if f is an indicator function?

1.4 Structure of some polygonal tilings in dimension 2

The one-dimensional tiling problem treated in the previous section is very particular. One cannot expect this rigid structure in higher dimension. For example, even when the tile is a square in two dimensions, one cannot expect every tiling of it to be fully-periodic, in the sense of posessing a period lattice of full-rank. One can, after all, make vertical columns of squares which can be shifted vertically, within themselves, arbitrarily, preserving the tiling property (see Figure 1 (a)). It is clear that there is no horizontal period here, in general. One might suspect that there is always, no matter what the tile, at least one period, but this phenomenon, if true, must happen only in dimension two. In dimension three one can construct cube tilings with no periods at all. First make horizontal layers of cubes some of which have no period along the x-axis and some others having no period along the y-axis. Consider these tiled slabs as rigid bodies and move each of them by an arbitrary horizontal vector thereby destroying all vertical periods as well.

Open Problem 2. *If $E \subseteq \mathbb{R}^2$, is it true that in any tiling $E + \Lambda = \mathbb{R}^2$ the set Λ must posess at least one period-vector?*

The main difficulty in dimension two and higher is that the zero set of $\widehat{f}$ is not a discrete set any more, at least under no set of reasonable assumptions about f (such as compact support was in dimension one). Therefore, from our basic condition (1.2) on obtains that $\widehat{\delta_\Lambda}$ is supported, in general, on a subset of the plane, which, under some reasonable assumptions, is a collection of submanifolds of codimension one. The structure of such distributions is much richer of course than those supported at points, and this is the main source of difficulty, at least compared with the one-dimensional problem.

In this section we will show the following result [**K00a**] in two-dimensions.

Theorem 1.8. (Kolountzakis, 2000)
Suppose that P is a symmetric convex polygon in the plane which tiles (multiply) with the multiset Λ:

$$P + \Lambda = m\mathbb{R}^2$$

at some integer level m. If P is not a parallelogram then Λ is a finite union of two-dimensional lattices.

The symmetry and convexity assumption here is only used to guarantee that each edge-direction appears in the polygon exactly twice. For a more general theorem see [**K00a**].

If one tries to use (1.2) directly, one encounters the problems mentioned above, mainly the fact that the zero set $\mathcal{Z}(\widehat{\chi_P})$ is not discrete, but rather a one-dimensional set.

Let e_1 and e_2 be two edges of the polygon P of the same direction u. By the symmetry of P they have the same length. We can then write (here e_1 and e_2 are viewed as point-sets in $\mathbb{R}^2$ and τ as a vector)

$$e_2 = e_1 + \tau,$$

for some $\tau \in \mathbb{R}^2$. (For each set A and vector x we write $A + x = \{a + x : a \in A\}$.) Let then μ_u be the measure which is equal to arc-length on e_1 and negative arc-length on e_2 (see Figure 7). Since every part of a translate of e_1 in the tiling $P + \Lambda$ has to be cancelled by part of a copy of e_2 it follows that

$$\sum_{\lambda\in\Lambda} \mu_u(x - \lambda)$$

is the zero measure in $\mathbb{R}^2$. It also intuitively obvious that the vanishing of the above measure for all relevant directions (i.e. those appearing as edge-directions) u also implies tiling at some integer level.

So a convex symmetric polygon P tiles multiply with a multiset Λ if and only if for each pair e and $e + \tau$ of parallel edges of P

$$\sum_{\lambda\in\Lambda} \mu_e(x - \lambda) = 0, \tag{1.10}$$

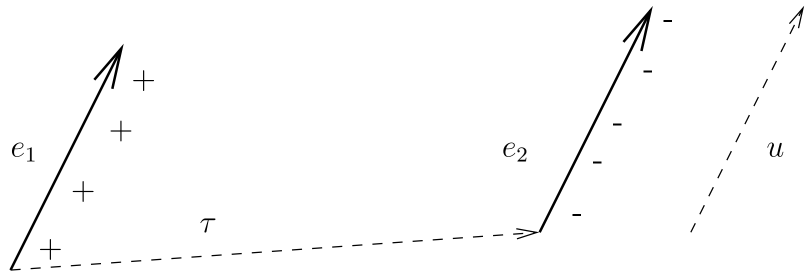

FIGURE 7: The measure μ_u supported on two parallel edges of the polygon e_1 and e_2, with opposite sign on each edge.

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where μ_e is the measure in $\mathbb{R}^2$ that is arc-length on e and negative arc-length on $e+\tau$. Condition (1.10) then becomes $\mu_e * \delta_\Lambda = 0$ or, taking Fourier Transforms (arguing as in §1.2),

$$\widehat{\mu_e} \cdot \widehat{\delta_\Lambda} = 0.$$

and

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \mathcal{Z}(\widehat{\mu_e}) \tag{1.11}$$

for *all* edge-directions e.

1.4.1 The shape of the zero-set

Here we study the zero-set of $\widehat{\mu_e}$ and determine its structure. We first calculate $\widehat{\mu_e}$ in the particular case when e is parallel to the x-axis, for simplicity. Let $\mu \in M(\mathbb{R}^2)$ be the measure defined by duality by

$$\mu(\phi) = \int_{-1/2}^{1/2} \phi(x,0)\ dx, \quad \forall \phi \in C(\mathbb{R}^2).$$

That is, μ is arc-length on the line segment joining the points $(-1/2, 0)$ and $(1/2, 0)$. Calculation gives

$$\widehat{\mu}(\xi,\eta) = \frac{\sin \pi\xi}{\pi\xi}.$$

Notice that $\widehat{\mu}(\xi,\eta) = 0$ is equivalent to $\xi \in \mathbb{Z} \setminus \{0\}$.

If μ_L is the arc-length measure on the line segment joining $(-L/2, 0)$ and $(L/2, 0)$ we have

$$\widehat{\mu_L}(\xi,\eta) = \frac{\sin \pi L\xi}{\pi\xi}$$

and

$$\mathcal{Z}(\widehat{\mu_L}) = \left\{(\xi,\eta) :\ \xi \in L^{-1}\mathbb{Z} \setminus \{0\}\right\}.$$

Write $\tau = (a, b)$ and let $\mu_{L,\tau}$ be the measure which is arc-length on the segment joining $(-L/2, 0)$ and $(L/2, 0)$ translated by $\tau/2$ and negative arc-length on the same segment translated by $-\tau/2$. That is, we have

$$\mu_{L,\tau} = \mu_L * (\delta_{\tau/2} - \delta_{-\tau/2}),$$

and, taking Fourier Transforms, we get

$$\widehat{\mu_{L,\tau}}(\xi,\eta) = -2\frac{\sin \pi L\xi}{\pi\xi} \sin \pi(a\xi + b\eta).$$

Define $u = \frac{\tau}{|\tau|^2}$ and $v = (1/L, 0)$. It follows that ($u^\perp$ is a unit vector orthogonal to u)

$$\mathcal{Z}(\widehat{\mu_{L,\tau}}) = (\mathbb{Z}u + \mathbb{R}u^\perp) \cup (\mathbb{Z} \setminus \{0\}v + \mathbb{R}v^\perp).$$

(Each of the two summands in the union above corresponds to each of the factors in the formula for $\widehat{\mu_{L,\tau}}$.) This a set of straight lines of direction $u^\perp$ spaced by $|u|$ and containing 0 plus a similar set of lines of direction $v^\perp$, spaced by $|v|$ and containing zero. However in the latter set of parallel lines the straight line through 0 has been removed (see Figure 8). We state this as a theorem for later use, formulated in a coordinate-free

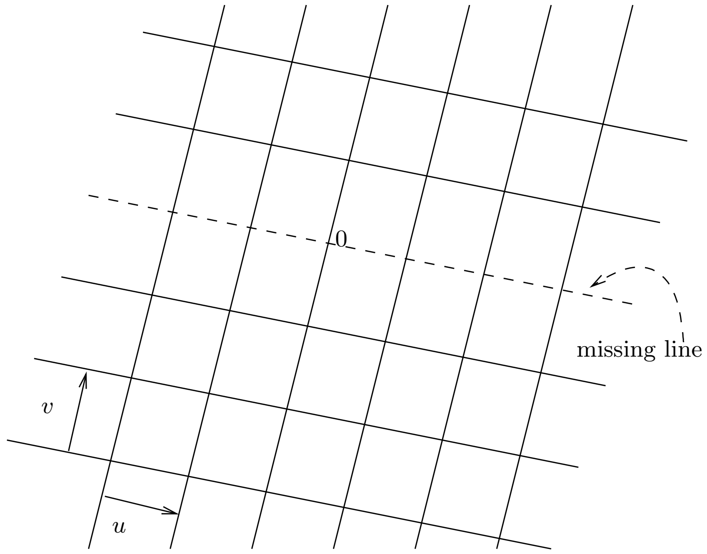


FIGURE 8: The zero-set $\mathcal{Z}(\widehat{\mu_{L,\tau}}) = (\mathbb{Z}u + \mathbb{R}u^\perp) \cup (\mathbb{Z} \setminus \{0\}v + \mathbb{R}v^\perp)$, with $u = \frac{\tau}{|\tau|^2}$ and $v = (1/L, 0)$

way.

Definition 1.5. (Geometric inverse of a vector)
The geometric inverse of a non-zero vector $u \in \mathbb{R}^2$ is the vector

$$u^* = \frac{u}{|u|^2}.$$

Theorem 1.9. *Let e and $e + \tau$ be two parallel line segments (translated by τ, of magnitude and direction described by e, symmetric with respect to 0). Let also $\mu_{e,\tau}$ be the measure which charges e with its arc-length and $e + \tau$ with negative its arc-length. Then*

$$\mathcal{Z}(\widehat{\mu_{e,\tau}}) = (\mathbb{Z}\tau^* + \mathbb{R}\tau^{*\perp}) \cup (\mathbb{Z} \setminus \{0\}e^* + \mathbb{R}e^{*\perp}). \tag{1.12}$$

1.4.2 Completion of the argument

The *intersection* of all the relevant $\mathcal{Z}(\widehat{\mu_e})$ is easily shown to be a discrete set, except when P is a parallelogram.

To conclude the argument we show that the tempered distribution $\widehat{\delta_\Lambda}$ is (a) locally a measure, and (b) the point masses of $\widehat{\delta_\Lambda}$ are uniformly bounded. This is accomplished using the following two Theorems.

Theorem 1.10. *Suppose that $\Lambda \in \mathbb{R}^d$ is a multiset with density ρ, $\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$, and that $\widehat{\delta_\Lambda}$ is a measure in a neighborhood of 0. Then $\widehat{\delta_\Lambda}(\{0\}) = \rho$.*

Proof. Take $\phi \in C^\infty$ of compact support with $\phi(0) = 1$. We have

$$\begin{aligned}
\widehat{\delta_\Lambda}(\{0\}) &= \lim_{t\to\infty} \widehat{\delta_\Lambda}(\phi(tx)) \\
&= \lim_{t\to\infty} \delta_\Lambda(t^{-d}\widehat{\phi}(\xi/t)) \\
&= \lim_{t\to\infty} t^{-d} \sum_{\lambda\in\Lambda} \widehat{\phi}(\lambda/t) \\
&= \lim_{t\to\infty} \sum_{n\in\mathbb{Z}^d} \sum_{\lambda\in Q_n} t^{-d}\widehat{\phi}(\lambda/t)
\end{aligned}$$

where, for fixed and large $T > 0$,

$$Q_n = [0,T)^d + Tn, \qquad n \in \mathbb{Z}^d.$$

Since Λ has density ρ it follows that for each $\epsilon > 0$ we can choose T large enough so that for all n

$$|\Lambda \cap Q_n| = \rho|Q_n|(1 + \delta_n),$$

with $|\delta_n| \le \epsilon$. For each n and $\lambda \in Q_n$ we have

$$\widehat{\phi}(\lambda/t) = \widehat{\phi}(Tn/t) + r_\lambda$$

with $|r_\lambda| \le CTt^{-1} \left\|\nabla\widehat{\phi}\right\|_{L^\infty(t^{-1}Q_n)}$. Hence

$$\begin{aligned}
\widehat{\delta_\Lambda}(\{0\}) &= \lim_{t\to\infty} \sum_{n\in\mathbb{Z}^d} t^{-d} \sum_{\lambda\in Q_n} (\widehat{\phi}(Tn/t) + r_\lambda) \\
&= \lim_{t\to\infty} \sum_{n\in\mathbb{Z}^d} t^{-d}\rho|Q_n|(1+\delta_n)\widehat{\phi}(Tn/t) + \\
&\quad \lim_{t\to\infty} \sum_{n\in\mathbb{Z}^d} t^{-d} \sum_{\lambda\in Q_n} r_\lambda \\
&= \lim_{t\to\infty} S_1 + \lim_{t\to\infty} S_2.
\end{aligned}$$

We have

$$\left| S_1 - \sum_n t^{-d}\rho|Q_n|\widehat{\phi}(Tn/t) \right| \le \epsilon \sum_n t^{-d}\rho|Q_n| \left|\widehat{\phi}(Tn/t)\right| \tag{1.13}$$

The first sum in (1.13) is a Riemann sum for $\rho\int_{\mathbb{R}^d}\widehat{\phi} = \rho$ and the second is a Riemann sum for $\rho\int_{\mathbb{R}^d}\left|\widehat{\phi}\right| < \infty$.

For S_2 we have

$$\begin{aligned}
|S_2| &\le C \sum_{n\in\mathbb{Z}^d} t^{-d}\rho|Q_n|(1+\delta_n)Tt^{-1}\left\|\nabla\widehat{\phi}\right\|_{L^\infty(t^{-1}Q_n)} \\
&\le C\rho Tt^{-1} \sum_{n\in\mathbb{Z}^d} t^{-d}|Q_n|\left\|\nabla\widehat{\phi}\right\|_{L^\infty(t^{-1}Q_n)}.
\end{aligned}$$

The sum above is a Riemann sum for $\int_{\mathbb{R}^d}\left|\nabla\widehat{\phi}\right|$, which is finite, hence $\lim_{t\to\infty} S_2 = 0$.

Since ϵ is arbitrary the proof is complete. ∎

Remark 1.3. *The same proof as that of Theorem 1.10 shows that, if*

$$\mu = \sum_{\lambda\in\Lambda} c_\lambda \delta_\lambda,$$

with $|c_\lambda| \le C$, Λ is of density 0 and the tempered distribution $\widehat{\mu}$ is locally a measure in the neighborhood of some point $a \in \mathbb{R}^2$, then we have $\widehat{\mu}(\{a\}) = 0$.

Theorem 1.11. *Suppose that the multiset $\Lambda \subset \mathbb{R}^d$ has density uniformly bounded by ρ and that, for some point $a \in \mathbb{R}^d$ and $R > 0$,*

$$\operatorname{supp} \widehat{\delta_\Lambda} \cap B_R(a) = \{a\}.$$

Then, in $B_R(a)$, we have $\widehat{\delta_\Lambda} = w\delta_a$, for some $w \in \mathbb{C}$ with $|w| \le \rho$.

Proof. It is well known that the only tempered distributions supported at a point a are finite linear combinations of the derivatives of δ_a. So we may assume that, for $\phi \in C^\infty(B_R(a))$,

$$\widehat{\delta_\Lambda}(\phi) = \sum_\alpha c_\alpha (D^\alpha \delta_a)(\phi) = \sum_\alpha (-1)^{|\alpha|} c_\alpha D^\alpha \phi(a), \tag{1.14}$$

where the sum extends over all values of the multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d \le m$ (the finite degree) and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ as usual.

We want to show that $m = 0$. Assume the contrary and let α_0 be a multiindex that appears in (1.14) with a non-zero coefficient and has $|\alpha_0| = m$. Pick a smooth function ϕ supported in a neighborhood of 0 which is such that for each multiindex α with $|\alpha| \le m$ we have $D^\alpha \phi(0) = 0$ if $\alpha \ne \alpha_0$ and $D^{\alpha_0}\phi(0) = 1$. (To construct such a ϕ, multiply the polynomial $(1/\alpha_0!)x^{\alpha_0}$ with a smooth function supported in a neighborhood of 0, which is identically equal to 1 in a neighborhood of 0.)

For $t \to \infty$ let $\phi_t(x) = \phi(t(x-a))$. Equation (1.14) then gives that

$$\widehat{\delta_\Lambda}(\phi_t) = t^m (-1)^m c_{\alpha_0}. \tag{1.15}$$

On the other hand, using

$$(\phi(t(x-a)))^\wedge(\xi) = e^{-2\pi i \langle a, \xi/t\rangle} t^{-d} \widehat{\phi}(\xi/t),$$

we get

$$\widehat{\delta_\Lambda}(\phi_t) = \sum_{\lambda \in \Lambda} e^{-2\pi i \langle a, \lambda/t\rangle} t^{-d} \widehat{\phi}(\lambda/t). \tag{1.16}$$

Notice that (1.16) is a bounded quantity as $t \to \infty$ by a proof similar to that of Theorem 1.10, while (1.15) increases like t^m, a contradiction.

Hence $\widehat{\delta_\Lambda} = w\delta_a$ in a neighborhood of a. The proof of Theorem 1.10 again gives that $|w| \le \rho$. ∎

We are now ready to prove the result [**K00a**] that finishes the argument.

Theorem 1.12. (Kolountzakis, 2000)
Suppose that $\Lambda \subset \mathbb{R}^2$ is a discrete multiset of uniformly bounded density and that

$$\widehat{\delta_\Lambda} = \left(\sum_{\lambda \in \Lambda} \delta_\lambda\right)^\wedge$$

is locally a measure with

$$\left|\widehat{\delta_\Lambda}\right|(B_R(0)) \le CR^2,$$

for some positive constant C and $R \ge 1$. Assume also that $\widehat{\delta_\Lambda}$ has discrete support. Then Λ is a finite union of translated lattices.

Proof. Define the sets (not multisets)

$$\Lambda_k = \{\lambda \in \Lambda : \ \lambda \text{ has multiplicity } k\}.$$

By Meyer's Theorem 1.5 (applied for the base set of the multiset Λ with the coefficients c_λ equal to the corresponding multiplicities) each of the Λ_k is in the coset ring of $\mathbb{R}^2$.

By Theorem 1.6 it follows that the discrete set Λ_k can be constructed from lattices in $\mathbb{R}^2$ (two-dimensional, one-dimensional or points) using finitely many operations and one shows easily that the set Λ_k has the form

$$\Lambda_k \;=\; \left(\bigcup_{j=1}^{J} A_j \;\setminus\; (B_1^{(j)} \cup \cdots \cup B_{n_j}^{(j)})\right) \;\cup\; \bigcup_{l=1}^{L} L_l \;\triangle\; F, \tag{1.17}$$

where $A_1, \ldots, A_J$ are 2-dimensional translated lattices, L_l and $B_i^{(j)}$ are 1-dimensional translated lattices and F is a finite set $(J, L \geq 0)$. The lattices A_j may be assumed to be have pairwise intersections of dimension at most 1.

We may thus write

$$\Lambda_k = A \triangle B, \tag{1.18}$$

with $A = \bigcup_{j=1}^{J} A_j$, where the 2-dimensional translated lattices A_j have pairwise intersections of dimension at most 1, and $\operatorname{dens} B = 0$.

Hence

$$\delta_{\Lambda_k} = \sum_{j=1}^{J} \delta_{A_j} + \mu,$$

where $\mu = \sum_{f \in F} c_f \delta_f$, $\operatorname{dens} F = 0$ and $|c_f| \leq C(J)$. The set F consists of B and all points contained in at least two of the A_j.

Combining for all k, and reusing the symbols A_j, μ and F, we get

$$\delta_{\Lambda} = \sum_{j=1}^{J} \delta_{A_j} + \mu.$$

But $\widehat{\delta_\Lambda}$ and $\sum_{j=1}^{J} \widehat{\delta_{A_j}}$ are both (by the assumption and the Poisson Summation Formula) discrete measures, and so is therefore $\widehat{\mu}$. However $\operatorname{dens} F = 0$ and the boundedness of the coefficients c_f implies that $\widehat{\mu}$ has no point masses (see Remark 1.3), which means that $\widehat{\mu} = 0$ and so is μ. Hence $\delta_\Lambda = \sum_{j=1}^{J} \delta_{A_j}$, or

$$\Lambda = \bigcup_{j=1}^{J} A_j, \quad \text{as multisets.}$$

∎

Last, observe that the support of $\widehat{\delta_\Lambda}$ is contained in the intersection of two grids of the type shown in Theorem 1.9, and has therefore (remember it's a discrete set) bounded density. This proves that $\left|\widehat{\delta_\Lambda}\right|(B_R(0)) \leq CR^2$ and we can invoke Theorem 1.12.

2 Lecture 2: Problems of lattice tiling.

Here we will examine several lattice tiling problems. The study of lattice tilings in Fourier space is particularly simple as explained in §1.2.1

$f + \Lambda$ is a tiling if and only if $\widehat{f}$ vanishes on the dual lattice Λ^*, except at zero.

The study of lattice tiling does not involve at all distributions which are not measures. The Fourier Analysis involved is nothing more than the usual multi-dimensional Fourier Series plus a change of variable to go from the integer to the arbitrary lattice.

2.1 A new equivalent form of a theorem of Hajós

Let us start by quoting a well known theorem of Minkowski in the Geometry of Numbers.

Theorem 2.1. (Minkowski, ca. 1900)
Let $A \in GL(d, \mathbb{R})$ have $\det A = 1$. Then there is $x \in \mathbb{Z}^d \setminus \{0\}$ with $\|Ax\|_\infty \le 1$.

Proof. Let $\Lambda = A\mathbb{Z}^d$ and $U = [-\frac{1}{2}, \frac{1}{2}]^d$. We want to show that $\Lambda \cap (2U)$ contains something besides 0. Suppose, on the contrary, that $\Lambda \cap (2U) = \{0\}$. Then, there is $\epsilon > 0$ such that for

$$U_\epsilon = [-\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon]^d$$

we have $\Lambda \cap (2U_\epsilon) = \{0\}$. We can rewrite this as

$$(\Lambda - \Lambda) \cap (U_\epsilon - U_\epsilon) = \{0\},$$

which means that the copies $U_\epsilon + \lambda$, $\lambda \in \Lambda$, are disjoint (we have a *packing*). But $\operatorname{dens} \Lambda = 1$ and $|U_\epsilon| > 1$, which is a contradiction, according to Lemma 1.3. ∎

The following theorem of Hajós [**Haj41**] proved a conjecture of Minkowski some forty years after it was posed. This conjecture concerned the case when one could have a strict inequality in Theorem 2.1.

Theorem 2.2. (Hajós, 1941)
Let $A \in GL(d, \mathbb{R})$ have $\det A = 1$. Then there is $x \in \mathbb{Z}^d$ with $\|Ax\|_\infty < 1$ unless A has an integral row.

Hajós actually worked on the following equivalent form of the Minkowski conjecture, which involves lattice tilings by a cube. This form was already known to Minkowski and most results on Minkowski's conjecture leading up to Hajós's eventual proof have used this form.

Theorem 2.3. *If $Q = [-1/2, 1/2]^d$ is a cube of unit volume in $\mathbb{R}^d$, $\Lambda \subset \mathbb{R}^d$ is a lattice, and*

$$\mathbb{R}^d = Q + \Lambda$$

is a lattice tiling of $\mathbb{R}^d$ then there are two cubes in the tiling that share a $(d-1)$-dimensional face. In other words, for some $i = 1, \ldots, d$, the standard basis vector $e_i = (0, \ldots, 0, 1, 0, \ldots, 0)^\top \in \Lambda$.

Keller [**Kel30**] conjectured that the same is true even without the lattice assumption. That is, Keller conjectured that in any tiling of Euclidean space by translates of a cube there are two cubes in the tiling which share a $(d-1)$-dimensional face. This is indeed true up to dimension 6 but was disproved by Lagarias and Shor [**LS92**] for $d \ge 10$. Mackey [**Mac02**] recently proved that Keller's conjecture is false in dimension 8 and higher. The only remaining case for which Keller's conjecture is still open is $d = 7$.

Theorem 2.2 $\Longrightarrow$ Theorem 2.3.

Let $\Lambda = A\mathbb{Z}^d$ with $\det A = 1$, $Q + \Lambda = \mathbb{R}^d$. Then, either there is a non-zero Λ-point in the interior of $2Q$ or A has an integral row. The first cannot happen because of the tiling assumption. Therefore $a_{ij} \in \mathbb{Z}$ for some i and for all j. Again because of tiling it follows that $\gcd(a_{i1}, \ldots, a_{i,d}) = 1$. Otherwise the i-th coordinates of all Λ-points would be multiples of $G = \gcd(a_{i1}, \ldots, a_{i,d}) > 1$, which is impossible (there would be gaps in the tiling). Let $\mathbb{R}^{d-1}$ be the subspace spanned by all e_j, $j \neq i$, and define $\Lambda' = \Lambda \cap \mathbb{R}^{d-1}$ and $Q' = Q \cap \mathbb{R}^{d-1}$. It follows that $\mathbb{R}^{d-1} = \Lambda' + Q'$ is a tiling of $\mathbb{R}^{d-1}$. By induction then Λ' contains some vector of the standard basis and so does Λ. ∎

Theorem 2.3 $\Longrightarrow$ Theorem 2.2.

Theorem 2.3 easily implies the seemingly stronger statement that, if $A\mathbb{Z}^d + Q = \mathbb{R}^d$ is a tiling then, after a permutation of the coordinate axes, the matrix A takes the form

$$\begin{pmatrix} 1 & 0 & 0 & \ldots & 0 \\ a_{2,1} & 1 & 0 & & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots \\ a_{d,1} & \ldots & \ldots & \ldots & 1 \end{pmatrix} \tag{2.1}$$

Using this remark, if $A\mathbb{Z}^d \cap (-1,1)^d = \{0\}$ we get, since $\det A = 1$, that $A\mathbb{Z}^d + Q = \mathbb{R}^d$ and, therefore, A is (after permutation of the coordinate axes) of the type (2.1), and thus has an integral row (and this property is preserved under permutation similarity). ∎

We now prove that the following is equivalent to Theorems 2.2 and 2.3 [**K98**].

Theorem 2.4. (Kolountzakis, 1998)
Let $B \in GL(d,\mathbb{R})$ have $\det B = 1$ and the property that for all $x \in \mathbb{Z}^d \setminus \{0\}$ some coordinate of the vector Bx is a non-zero integer. Then B has an integral row.

Open Problem 3. *Prove this combinatorial statement directly, thereby obtaining a new proof of the Minkowski Conjecture.*

Remark 2.1. *One might think that Theorem 2.4 can be proved equivalent directly to Theorem 2.2, which it resembles most. It is, indeed, clear that Theorem 2.2 implies Theorem 2.4. However, the proof that is given here is that of the equivalence of Theorems 2.4 and 2.3. I do not know of a more direct proof of the fact that Theorem 2.4 implies Theorem 2.2.*

We shall need the following simple lemma.

Lemma 2.1. *Let $A \in GL(d,\mathbb{R})$ be a non-singular matrix. The lattice $A^{-\top}\mathbb{Z}^d$ contains the basis vector e_i if and only if the i-th row of A is integral.*

Proof. Without loss of generality assume $i = 1$.

If $e_1 \in A^{-\top}\mathbb{Z}^d$ then $e_1 = A^{-\top}x$ for some $x \in \mathbb{Z}^d$. Therefore, for all $y \in \mathbb{Z}^d$ we have

$$(Ay)_1 = e_1^\top Ay = x^\top A^{-1}Ay = x^\top y \in \mathbb{Z}.$$

It follows that $(Ay)_1 \in \mathbb{Z}$ for all $y \in \mathbb{Z}^d$ and the first row of A is integral.

Conversely, if the first row of A is integral, then, for all $y \in \mathbb{Z}^d$

$$\mathbb{Z} \ni (Ay)_1 = x^\top y,$$

where $A^{-\top}x = e_1$ ($x \in \mathbb{R}^d$). It follows that $x \in \mathbb{Z}^d$ and $e_1 \in A^{-\top}\mathbb{Z}^d$. ∎

Proof of the equivalence of Theorems 2.3 and 2.4.

Let $f(x) = \mathbf{1}(x \in Q)$ be the indicator function of the unit-volume cube $Q = [-1/2, 1/2]^d$. A simple calculation shows that

$$\widehat{f}(\xi) = \prod_{j=1}^{d} \frac{\sin \pi\xi_j}{\pi\xi_j}, \tag{2.2}$$

so that

$$\mathcal{Z} := \left\{ \widehat{f} = 0 \right\} = \left\{ \xi \in \mathbb{R}^d : \text{ some } \xi_j \text{ is a non-zero integer} \right\}. \tag{2.3}$$

Therefore, if $\Lambda = B^{-\top}\mathbb{Z}^d$ then (since Λ has volume 1)

$$Q + \Lambda = \mathbb{R}^d \Longleftrightarrow \Lambda^* \setminus \{0\} \subseteq \mathcal{Z},$$

where $\Lambda^* = B\mathbb{Z}^d$. In words, Q tiles with $\Lambda = B^{-\top}\mathbb{Z}^d$ if and only if for every $x \in \mathbb{Z}^d \setminus \{0\}$ the vector Bx has some non-zero integral coordinate.

Theorem 2.3 $\Longrightarrow$ Theorem 2.4.

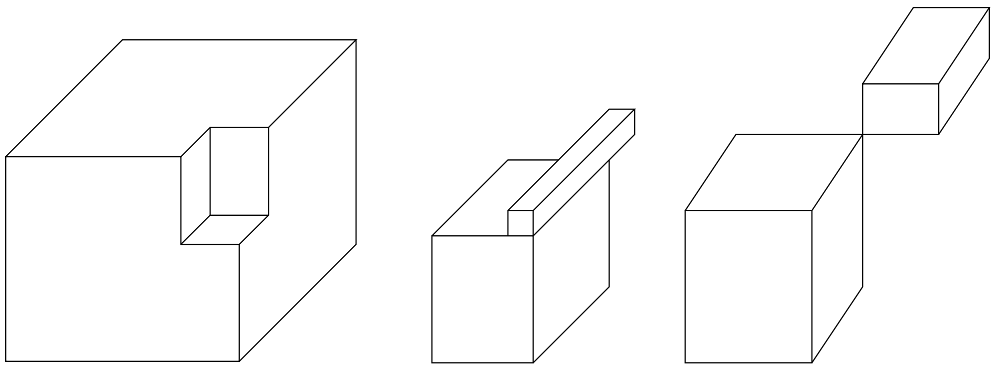
Suppose $x \in \mathbb{Z}^d \setminus \{0\}$ implies some $(Bx)_i \in \mathbb{Z} \setminus \{0\}$. Then $Q + \Lambda = \mathbb{R}^d$ and from Theorem 2.3, say, $e_1 \in \Lambda$, which, from Lemma 2.1, implies that the first row of B is integral.

Theorem 2.4 $\Longrightarrow$ Theorem 2.3.

Assume $Q + \Lambda = \mathbb{R}^d$. It follows that for every $x \in \mathbb{Z}^d \setminus \{0\}$ the vector Bx has some non-zero integral coordinate. By Theorem 2.4 B must have an integral row, which, by Lemma 2.1, implies that some $e_i \in \Lambda$. ∎

2.2 Tilings by notched and extended cubes

In this section we prove that some simple shapes (like those in Figure 9) admit lattice tilings.



(a) (b) (c)

FIGURE 9: These shapes admit lattice tilings

2.2.1 The notched cube

We consider first the unit cube

$$Q = \left[-\frac{1}{2}, \frac{1}{2}\right]^d$$

from whose corner (say in the positive orthant) a rectangle R has been removed with sides-lengths $\delta_1, \ldots, \delta_d$ ($0 \le \delta_j \le 1$). That is, we consider the "notched cube":

$$N = Q \setminus R$$

where

$$R = \prod_{j=1}^{d} \left[\frac{1}{2} - \delta_j, \frac{1}{2}\right].$$

It is shown in Figure 9 (a).

We give a new [**K98**], Fourier-analytic, proof of the following result of Stein [**St90**].

Theorem 2.5. (Stein, 1990)
The notched cube N admits a lattice tiling of $\mathbb{R}^d$.

After a simple calculation we obtain

$$\widehat{\chi_N}(\xi) = \prod_{j=1}^d \frac{\sin \pi\xi_j}{\pi\xi_j} - F(\xi) \prod_{j=1}^d \frac{\sin \pi\delta_j\xi_j}{\pi\xi_j}, \tag{2.4}$$

where $F(\xi) = \exp(\pi i K(\xi))$ with

$$K(\xi) = \sum_{j=1}^d (\delta_j - 1)\xi_j. \tag{2.5}$$

Using (1.5) it is enough to exhibit a lattice $\Lambda \subset \mathbb{R}^d$, of volume equal to

$$|N| = 1 - \delta_1 \cdots \delta_d,$$

such that $\widehat{\chi_N}$ vanishes on $\Lambda^* \setminus \{0\}$.

2.2.2 Lattices in the zero-set

We define the lattice Λ^* as those points ξ for which

$$\begin{aligned} \xi_1 - \delta_2\xi_2 &= n_1, \\ \xi_2 - \delta_3\xi_3 &= n_2, \\ &\cdots \\ \xi_d - \delta_1\xi_1 &= n_d, \end{aligned} \tag{2.6}$$

for some $n_1, \ldots, n_d \in \mathbb{Z}$. That is, $\Lambda^* = A^{-1}\mathbb{Z}^d$, where

$$A = \begin{pmatrix} 1 & -\delta_2 & & & \\ & 1 & -\delta_3 & & \\ & & \ddots & & \\ & & & 1 & -\delta_d \\ -\delta_1 & & & & 1 \end{pmatrix}. \tag{2.7}$$

Therefore $\Lambda = A^\top \mathbb{Z}^d$ and the volume of Λ is equal to $|\det A|$. Expanding A along the first column we get easily that $\det A = 1 - \delta_1 \cdots \delta_d$, which is the required volume.

We now verify that $\widehat{\chi_N}$ vanishes on $\Lambda^* \setminus \{0\}$.

Assume that $0 \neq \xi \in \Lambda^*$. Adding up the equations in (2.6) we get

$$K = K(\xi) = -(n_1 + \cdots + n_d).$$

If all the coordinates of ξ are non-zero we can write

$$\widehat{\chi_N}(\xi) = \frac{1}{\pi^d \xi_1 \cdots \xi_d} \left(\prod_{j=1}^d \sin \pi\xi_j - (-1)^{K(\xi)} \prod_{j=1}^d \sin \pi\delta_j\xi_j \right). \tag{2.8}$$

Observe from (2.6) that

$$\sin \pi\xi_j = (-1)^{n_j} \sin \pi\delta_{j+1}\xi_{j+1},$$

where the subscript arithmetic is done modulo d, from which we get $\widehat{\chi_N}(\xi) = 0$, since the factors in the two terms of (2.8) match one by one.

It remains to show that $\widehat{\chi_N}(\xi) = 0$ even when ξ has some coordinate equal to 0, say $\xi_1 = 0$.

Consider the numbers $\xi_1, \ldots, \xi_d$ arranged in a cycle and let

$$I = \{\xi_m, \xi_{m+1}, \ldots, \xi_1, \ldots, \xi_{k-1}, \xi_k\}$$

be an interval around ξ_1 which is maximal with the property that all its elements are 0. Then $\xi_{m-1} \neq 0$ and $\xi_{k+1} \neq 0$ and from (2.6) we get

$$\xi_{m-1} - \delta_m \xi_m = n_m \quad \text{and} \quad \xi_k - \delta_{k+1}\xi_{k+1} = n_k. \tag{2.9}$$

We deduce that n_m and n_k are both non-zero and therefore that ξ_{m-1} and $\delta_{k+1}\xi_{k+1}$ are both non-zero integers and $\sin \pi\xi_{m-1} = \sin \pi\delta_{k+1}\xi_{k+1} = 0$. This means that both terms in (2.4) vanish and so does $\widehat{\chi_N}(\xi)$. ∎

So we proved that for the lattice $\Lambda = A^\top \mathbb{Z}^d$, where A is defined in (2.7), we have $N + \Lambda = \mathbb{R}^d$. Clearly, if σ is a cyclic permutation of $\{1, \ldots, d\}$ and if instead of the matrix A we have the matrix A' whose i-th row has 1 on the diagonal, $-\delta_{\sigma i}$ at column σi and 0 elsewhere, we get again a lattice tiling with the lattice $(A')^\top \mathbb{Z}^d$. Stein [**St90**] as well as Schmerl [**Sch94**] have shown that these $(d-1)!$ lattice tilings of the notched cube (one for each cyclic permutation of $\{1., \ldots, d\}$) are all non-isometric when the side-lengths δ_j are all distinct.

A deeper result of Schmerl [**Sch94**] is that there are no other translational tilings of the notched cube, lattice or not. This is something that cannot apparently be proved with the Fourier Analysis approach.

2.2.3 Extended cubes

Let us now allow the parameters $\delta_1, \ldots, \delta_d$ to take on any non-zero real value subject only to the restriction

$$\delta_1 \cdots \delta_d \neq 1, \tag{2.10}$$

and let the function $\varphi(\xi)$ be equal to the right-hand side of (2.4). Let again the matrix A be defined by (2.7) and $\Lambda = A^\top \mathbb{Z}^d$ as before. We have again $\det A = 1 - \delta_1 \cdots \delta_d$.

The calculations we did in §2.2.2 show that φ vanishes on $\Lambda^* \setminus \{0\}$, hence, if $\check{\varphi}$ is the inverse Fourier Transform of φ, $\check{\varphi}$ tiles $\mathbb{R}^d$ with Λ and weight

$$\frac{\varphi(0)}{|1 - \delta_1 \cdots \delta_d|} = \operatorname{sgn}(1 - \delta_1 \cdots \delta_d), \tag{2.11}$$

where $\operatorname{sgn}(x) = \pm 1$ is the sign of x.

The function $\check{\varphi}$ is given by

$$\check{\varphi}(x) = \chi_Q(x) - \operatorname{sgn}(\delta_1 \cdots \delta_d)\psi(x), \tag{2.12}$$

where

$$\psi(x) = \chi_Q\left(\frac{x_1 - (1-\delta_1)/2}{|\delta_1|}, \ldots, \frac{x_d - (1-\delta_d)/2}{|\delta_d|}\right). \tag{2.13}$$

Notice that $\psi(x)$ is the indicator function of a rectangle $R = R(\delta_1, \ldots, \delta_d)$ with side-lengths $|\delta_1|, \ldots, |\delta_d|$ centered at the point

$$P = \left(\frac{1}{2}, \ldots, \frac{1}{2}\right) - \frac{1}{2}(\delta_1, \ldots, \delta_d). \tag{2.14}$$

The rectangle R intersects the interior of Q only in the case $\delta_1 > 0, \ldots, \delta_d > 0$ and when this happens $\check{\varphi}$ is an indicator function only if we also have $\delta_1 \leq 1, \ldots, \delta_d \leq 1$, which is the case of the notched cube that we examined in §2.2.2.

Otherwise (not all the δs are non-negative) $\check{\varphi}$ is an indicator function only when $\mathrm{sgn}(\delta_1 \cdots \delta_d) = -1$, i.e., the number of negative δs is odd. In this case we have that

$$\check{\varphi} = \chi_{Q \cup R}$$

and from (2.11) we get that $Q \cup R$ tiles with Λ and weight 1. We can now prove the following [**K98**].

Theorem 2.6. (Kolountzakis, 1998)
Let Q and R be two axis-aligned rectangles in $\mathbb{R}^d$ with sides of arbitrary length and disjoint interiors. Assume also that Q and R have a vertex K in common and intersection of odd codimension.

Then $Q \cup R$ admits a lattice tiling of $\mathbb{R}^d$ of weight 1.

For example, the extended cubes shown in Figure 9 (b),(c) admit lattice tilings of $\mathbb{R}^3$, as the corresponding codimensions are 1 and 3.

Proof. After a linear transformation we can assume that $Q = [-1/2, 1/2]^d$, that Q and R share the vertex $K = (1/2, \ldots, 1/2)$ and that $Q \cap R$ has codimension k (an odd number) and

$$Q \cap R \;\subseteq\; \left\{ x \in Q : \; x_1 = \cdots = x_k = \frac{1}{2} \right\}.$$

Let the side-lengths of R be $\gamma_1, \ldots, \gamma_d > 0$. Define

$$\delta_j = \begin{cases} -\gamma_j, & \text{if } 1 \le j \le k, \\ \gamma_j, & \text{if } k+1 \le j \le d. \end{cases}$$

It follows that, with this assignment for the δ_j, the indicator function of R is equal to the function $-\mathrm{sgn}(\delta_1 \cdots \delta_d)\psi(x)$ of (2.12) and tiling follows from the previous discussion. ∎

Most likely the extended cubes with an intersection of even codimension do not tile, at least not for general side-lengths. This is clear in dimension two and it is conceivable that some combinatorial argument could easily show this in any dimension. The Fourier Analysis approach does not seem to be very helpful when one tries to disprove that something is a translational tile.

Open Problem 4. *In the setting of Theorem 2.6 prove that if the codimension is even then the set $Q \cup R$ is not a tile.*

2.3 The Steinhaus tiling problem

2.3.1 The original, two-dimensional case

Steinhaus [**Mos81**, problem 59] asked whether there is a planar set S which, no matter how translated and rotated, always contains exactly one point with integer coordinates.

Definition 2.1. (Steinhaus property)
A set $S \subset \mathbb{R}^2$ has the Steinhaus property *if for every $x \in \mathbb{R}^2$ and for every rotation*

$$A_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

we have

$$\#\left(\mathbb{Z}^2 \cap (A_\theta S + x)\right) = 1, \tag{2.15}$$

where $A_\theta S + x = \{A_\theta s + x : \; s \in S\}$.

Sierpiński [**Sie59**] first proved that a set which is bounded and either open or closed cannot have the Steinhaus property. Croft [**Cro82**] and Beck [**Bec89**] proved the same of any set which is bounded and measurable. (Croft's approach is more direct and geometric. Beck is using Fourier Analysis.) Ciucu [**Ciu96**] shows that any Steinhaus set must have empty interior, without assuming boundedness. Several variations of the problem have been investigated by Komjáth [**Kom92**] from a rather different point a view, where one places a different subgroup of the plane in place of $\mathbb{Z}^2$.

Very recently it was shown by Jackson and Mauldin [**JM02**] that Steinhaus sets do indeed exist. But the construction there does not furnish measurable such sets and it is precisely under the assumption of measurability that we study the existence problem for Steinhaus sets here, using Fourier Analysis.

To begin, notice that the question of Steinhaus can be rephrased as follows:

(a) Is there a set E which tiles the plane if translated at any rotated copy of $\mathbb{Z}^2$?

(b) Or, is there a common set of coset representatives (fundamental domain) of all groups $R_\theta\mathbb{Z}^2$ in the group $\mathbb{R}^2$?

We only care for measurable Steinhaus sets (if they exist) so tiling, above, is to be interpreted in the almost everywhere sense, as it is normally interpreted throughout this survey.

As first noticed by Beck [**Bec89**], the Steinhaus question in the form (a), above, is equivalent to asking if there exists a measurable set $E \subseteq \mathbb{R}^2$, of measure 1, such that the Fourier Transform of its indicator function vanishes on all circles of the plane which are centered at the origin and pass through some point of the integer lattice $\mathbb{Z}^2$. This is so since for a set to have the Steinhaus property it must tile the plane when translated by any rotation of $\mathbb{Z}^2$ (this alone implies of course that $|E| = 1$). These sets are lattices, hence this is equivalent to $\widehat{\chi_E}$ vanishing on all these lattices, which are self-dual. The union of these rotated lattices is precisely the set of circles mentioned above. We state this as a Theorem.

Theorem 2.7. *A measurable set $E \subseteq \mathbb{R}^2$ is simultaneously a tile for all rotations of $\mathbb{Z}^2$ if and only if it has measure 1 and its Fourier Transform $\widehat{\chi_E}$ vanishes on all circles with center at the origin and radius of the form $\sqrt{m^2+n^2}$, with $m, n \in \mathbb{N}$, not both 0.*

It is now easy to see that such sets cannot be bounded, if they exist. Indeed, the restriction onto any line L through 0 of $\widehat{\chi_E}$ is nothing but the one-dimensional Fourier Transform of the function χ_E projected onto L, i.e., of the function

$$f(t) = \int_{L^\perp} \chi_E(tu+s)\, ds,$$

where u is a unit vector on L and $L^\perp$ is the line through 0 which is orthogonal to L. But if E is bounded the function $f(t)$ has compact support, hence $\widehat{\chi_E}(tu)$ is an entire function of exponential type, and, as such, it should have at most $C \cdot R$ zeros in the interval $(-R, R)$, where $C > 0$ is a constant. (See the discussion in §1.3.2.) However, the number of zeros of $\widehat{\chi_E}(tu)$ is twice the number of circles out to radius R, or, in other words, twice the number of integers expressible as a sum of two integer squares and of size up to R^2. But this number is almost quadratic in R. It is a well known result of Landau [**Fri82**] that it is $\sim cR^2 \log^{-1/2} R$.

With a more careful and quantitative approach along similar lines, but not using entire functions, it was then proved by the author [**K96**] that any set E with the Steinhaus property must be large at infinity:

$$\int_E |x|^\alpha\, dx = \infty, \quad \text{for any } \alpha > \tfrac{10}{3}.$$

With much more care it was obtained in [**KW99**] by the author and Tom Wolff that

Theorem 2.8. (Kolountzakis and Wolff, 1997)
If $E \subseteq \mathbb{R}^2$ is a measurable Steinhaus set then $\int_E |x|^\alpha = \infty$, for all $\alpha > 46/27$.

The number 46/27 comes from the best known estimate known for the *circle problem.* This is the problem where one asks for the best upper estimates in the error term $E(R)$ (as $R \to \infty$) in the expression

$$N(R) = \pi R^2 + E(R),$$

where $N(R)$ is the number of integer lattice points in the disk $\{|x| \le R\} \subseteq \mathbb{R}^2$. Even if the conjectured best possible upper bound $E(R) = O(R^{1/2+\epsilon})$ gets proved the estimate for the Steinhaus tiling problem in Theorem 2.8 would only become true for all $\alpha > 1$. So it appears that if one is going to disprove the existence of measurable Steinhaus sets in dimension two one needs some rather different approach.

This seems to be the state of knowledge for the two-dimensional case.

2.3.2 The problem in dimension $d \ge 3$

The Steinhaus problem generalizes very naturally to any dimension. One asks for a set $E \subseteq \mathbb{R}^d$ such that no matter what orthogonal linear transformation you apply to it, it still tiles $\mathbb{R}^d$ when translated by $\mathbb{Z}^d$. With precisely the same reasoning as before, one is looking for a measurable set of measure 1 such that the Fourier Transform of its indicator function vanishes on all spheres centered at the origin that contain some integer lattice point.

It is because of the fact that we know precisely which numbers are representable as sums of three squares that the following result [**KW99**] holds.

Theorem 2.9. (Kolountzakis and Wolff, 1997)
If $f \in L^1(\mathbb{R}^d)$, $d \ge 3$, and $\widehat{f}$ vanishes on all spheres centered at the origin through some lattice point, then f is a.e. equal to a continuous function.

In particular, there are no measurable Steinhaus sets in dimension $d \ge 3$.

Here we show an alternative way [**KP02**] of proving that there are no sets with the Steinhaus property in dimension $d \ge 3$. We emphasize though that Theorem 2.9 is much stronger than Theorem 2.10 given below. See also some related results of Mauldin and Yingst [**MY02**].

Theorem 2.10. (Kolountzakis and Papadimitrakis, 2000)
There are no measurable Steinhaus sets in dimension $d \ge 3$.

Proof. In any dimension d write $\mathcal{B}$ for the union of all spheres centered at the origin that go through at least one lattice point. The point 0 is included in $\mathcal{B}$.

Assume from now on that the set E is a Steinhaus set in dimension d.

Suppose now that we can find a lattice $\Lambda^* \subset \mathcal{B}$ with $\det \Lambda^*$ not an integer. Since $\widehat{\chi_E}$ vanishes on $\Lambda^* \setminus \{0\}$ it follows that $E + \Lambda$ is a tiling at level $\ell = |E| \times \operatorname{dens} \Lambda = 1 \times \det \Lambda^*$, which is not an integer. This is a contradiction as, obviously, any set may only tile at an integral level.

Looking at the quadratic form $\langle A^\top A x, x \rangle$ for each lattice $\Lambda^* = A\mathbb{Z}^d$ we summarize the above observations in the following lemma

Lemma 2.2. *If there exists a positive definite quadratic form $Q(x) = Q(x_1, \ldots, x_d) = \langle Bx, x \rangle$ such that for all integral $x_1, \ldots, x_d$ its value is the sum of d integer squares, and the determinant of Q, $\det B$, is not the square of an integer, then there are no Steinhaus sets in dimension d.*

The case $d \ge 4$:
Consider the symmetric 4×4 matrix B with 1 on the diagonal and 1/2 everywhere else. The matrix B is positive definite (its eigenvalues are 1/2, 1/2, 1/2 and 5/2) and its determinant is 5/16. It defines the quadratic form

$$Q(x) = Q(x_1, \ldots, x_4) = \langle Bx, x \rangle = \sum_{i=1}^{4} x_i^2 + \sum_{i>j} x_i x_j,$$

which is obviously integer valued and has non-square determinant. Furthermore, every non-negative integer may be written as a sum of four squares (Lagrange). From Lemma 2.2 it follows that there are no Steinhaus sets for $d = 4$. We easily see that this extends to all higher dimensions by taking as our matrix the identity in one corner of which sits the 4×4 matrix B described above.

The case $d = 3$:
The determinant of the form that appears in the following Theorem is $2 \cdot 11 \cdot 6$, which is not a square, hence there are no Steinhaus sets in dimension 3.

Theorem 2.11. *For each $x, y, z \in \mathbb{Z}$ the number*

$$Q(x, y, z) = 2x^2 + 11y^2 + 6z^2$$

is a sum of three integer squares.

Proof. Suppose this is false and that there are $(x_0, y_0, z_0) \neq (0, 0, 0)$ and

(a) $Q(x_0, y_0, z_0)$ is not a sum of three squares, and

(b) $x_0^2 + y_0^2 + z_0^2$ is minimal.

From (a), and the well known characterization of those natural numbers that cannot be written as a sum of three squares, we have that

$$Q(x_0, y_0, z_0) = 4^\nu(8k + 7), \quad \nu \geq 0, k \geq 0.$$

If all x_0, y_0, z_0 are even, we have $\nu \geq 1$, and, setting $x_0 = 2x_1$, $y_0 = 2y_1$ and $z_0 = 2z_1$, we obtain that $Q(x_1, y_1, z_1)$ is not a sum of three squares, which contradicts the minimality of the initial triple (x_0, y_0, z_0). We conclude that at least one of x_0, y_0, z_0 is odd.

Case No 1: $\nu = 0$.

Then $Q(x_0, y_0, z_0) = 7 \bmod 8$. But the quadratic residues mod 8 are 0, 1 and 4, and one checks by examining all the possibilities that Q is never 7 mod 8.

Case No 2: $\nu = 1$.

Then $Q(x_0, y_0, z_0) = 32k + 28$. Hence y_0 is even, say $y_0 = 2y_1$. We get

$$x_0^2 + 22y_1^2 + 3z_0^2 = 16k + 14,$$

from which we conclude that x_0 and z_0 are odd, $x_0 = 2x_1 + 1$, $z_0 = 2z_1 + 1$. Substitution gives

$$\begin{aligned}
4x_1^2 + 4x_1 + 1 + 22y_1^2 + 12z_1^2 + 12z_1 + 3 &= 16k + 14 \\
2x_1(x_1 + 1) + 11y_1^2 + 6z_1(z_1 + 1) + 2 &= 8k + 7 \\
2x_1(x_1 + 1) + 11y_1^2 + 6z_1(z_1 + 1) &= 5 \bmod 8.
\end{aligned}$$

But $\xi^2 + \xi = 0$ or 2 or 4 or 6 mod 8, for all ξ, hence, by applying this to the first and last term in the above sum, and checking all possibilities we get a contradiction.

Case No 3: $\nu \geq 2$.

As in Case No 2: $y_0 = 2y_1$, $z_0 = 2z_1 + 1$, $x_0 = 2x_1 + 1$. Hence

$$2x_1(x_1 + 1) + 11y_1^2 + 6z_1(z_1 + 1) + 2 = 4^{\nu-1}(8k + 7), \quad \nu - 1 \geq 1.$$

So y_1 is even, $y_1 = 2y_2$, which gives

$$x_1(x_1 + 1) + 22y_2^2 + 3z_1(z_1 + 1) + 1 = 2 \cdot 4^{\nu-2}(8k + 7),$$

a contradiction as the left hand side is odd while the right hand side is even. ∎

We point out here that the actual quadratic form was only found by a semi-automated computer search. See [**MY02**] for a more systematic study of the method.

It is also shown in [**KP02**] that the method shown above cannot be applied in dimension 2 to show the non-existence of measurable sets with the Steinhaus property.

Theorem 2.12. (Kolountzakis and Papadimitrakis, 2002)
Any positive-definite binary quadratic form whose values are always sums of two integer squares must have a determinant which is the square of an integer.

2.4 Multi-lattice tiles

2.4.1 A "finite" Steinhaus problem

The Steinhaus question essentially asks if there is a set in the plane which is *simultaneously* a translational tile for each translation set in the collection

$$\left\{R_\theta \mathbb{Z}^2 :\ 0 \le \theta < 2\pi\right\}, \quad (R_\theta \text{ is rotation by } \theta).$$

Restricting ourselves to the measurable case again it is easy to see, using, for example, the Fourier method, that it is sufficient for a set to be a tile for a countable dense (in the obvious sense) subset of these lattices (groups) in order to be a tile for all of them.

The problem only becomes significantly different if one restricts oneself to a *finite* collection of lattices

$$\Lambda_0, \ldots, \Lambda_n \subseteq \mathbb{R}^d,$$

all of the same volume, say volume 1, and asks for a measurable subset of $\mathbb{R}^d$ which tiles with all of them. It turns out [**K97**] that this is generically feasible and we give here a construction.

Theorem 2.13. (Kolountzakis, 1997)
If the lattices $\Lambda_0, \ldots, \Lambda_n \subset \mathbb{R}^d$ all have the same volume and if the sum of their dual lattices

$$\Lambda_0^* + \cdots + \Lambda_n^*$$

is a direct sum (i.e. there are no non-trivial relations $\lambda_0 + \cdots + \lambda_n = 0$ with $\lambda_i \in \Lambda_i^*$) then they possess a Borel measurable common tile (which is generally unbounded).

Proof. The common tile $\Omega \subseteq \mathbb{R}^d$ that we construct is a countable union of disjoint closed polyhedra (in fact, rectangles).

Definition 2.2. (Property A)
We shall say that a collection of lattices $\Lambda_0, \ldots, \Lambda_n \subset \mathbb{R}^d$ has Property A if for each $\epsilon > 0$ and for each $x_0, \ldots, x_n \in \mathbb{R}^d$ there exist $\lambda_0 \in \Lambda_0, \ldots, \lambda_n \in \Lambda_n$, with $|\lambda_j|$ arbitrarily large, such that

$$|x_i - \lambda_i - (x_j - \lambda_j)| \le \epsilon, \quad \textit{for all } i, j = 0, \ldots, n. \tag{2.16}$$

That is, we can get any collection of points $x_0, \ldots, x_n \in \mathbb{R}^d$ arbitrarily close to each other by translating x_i by some $\lambda_i \in \Lambda_i$, $i = 0, \ldots, n$.

We first show that if the given collection of lattices has Property A then it has a common tile. At the end of the proof we indicate why it is precisely the collections of lattices with their duals having a direct sum that have Property A.

The letter C will stand in this section for a positive constant that may not depend on the parameter $K \to \infty$ and this constant is not necessarily the same in all its occurences.

The lattices Λ_j, $j = 0, \ldots, n$, are given by

$$\Lambda_j = A_j \mathbb{Z}^d, \quad \det A_j = 1. \tag{2.17}$$

Let D_j be the standard tile for the lattice Λ_j, i.e.,

$$D_j = A_j [0,1)^d, \tag{2.18}$$

which is a parallelepiped of volume 1.

Let $\Omega_0 = \emptyset$. In the end we shall have

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k,$$

where the K-th approximation

$$A_K = \bigcup_{k=1}^{K} \Omega_k$$

has measure $\mu(A_K) \to 1$, as $K \to \infty$, and for each $j = 0, \ldots, n$ almost all cosets $x + \Lambda_j$ have no more than one point in A_K. It follows that Ω contains exactly one element from almost all the cosets of Λ_j, for each $j = 0, \ldots, n$, and is therefore a common tile for the collection $\Lambda_0, \ldots, \Lambda_n$. Assume that we have already defined $\Omega_0, \ldots, \Omega_K$. The set Ω_{K+1} will be defined as follows. The "projection" $\pi_j : \mathbb{R}^d \to D_j$ is defined by the relation

$$x - \pi_j(x) \in \Lambda_j.$$

The "leftover" after stage K is then defined by

$$L_j^{(K)} = D_j \setminus \pi_j(A_K), \quad \text{for } j = 0, \ldots, n. \tag{2.19}$$

We have to ensure that $\mu\left(L_j^{(K)}\right) \to 0$, as $K \to \infty$.

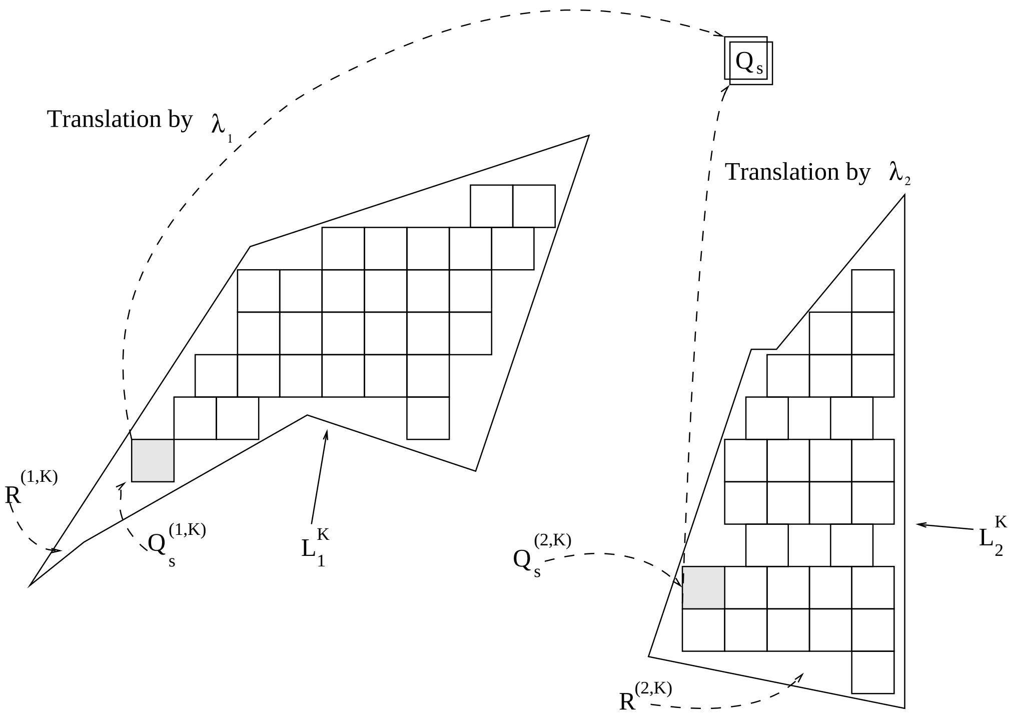


FIGURE 10: Construction of the common tile for two lattices, $d = 2$

Our construction will guarantee that each of the leftovers $L_j^{(K)}$ consists of a finite collection of polyhedra. Choose $\epsilon > 0$ to be so small so as to be able to write

$$L_j^{(K)} = \left(\bigcup_{s=1}^{S} Q_s^{(j,K)} \right) \cup R^{(j,K)}, \quad (j = 0, \ldots, n) \tag{2.20}$$

where the $Q_s^{(j,K)}$, $s = 1, \ldots, S = S(K)$, are axis-aligned, closed cubes with disjoint interiors of side ϵ, and

$$\mu\left(R^{(j,K)}\right) \le \frac{1}{K}. \tag{2.21}$$

Notice that the same number $S = S(K)$ of cubes is used independently of j. (The conctruction is shown for two lattices in Figure 10 in dimension $d = 2$.)

For each $s = 1, \ldots, S$, let $c_s^{(j,K)}$ be the center of the cube $Q_s^{(j,K)}$ and, using Property A, define $\lambda_s^{(j,K)} \in \Lambda_j$ to be such that all

$$c_s^{(j,K)} - \lambda_s^{(j,K)}, \quad j = 0, \ldots, n,$$

are at most $\frac{\epsilon}{K}$ apart. The $\lambda_s^{(j,K)}$ are also taken large enough so that, for fixed j, no two translated cubes $Q_s^{(j,K)} - \lambda_s^{(j,K)}$ overlap.

Consider then the intersection of the $n+1$ translated cubes

$$\widetilde{Q}_s^{(K)} = \bigcap_{j=0}^{n} \left(Q_s^{(j,K)} - \lambda_s^{(j,K)} \right) \tag{2.22}$$

and notice that

$$\mu(\widetilde{Q}_s^{(K)}) \geq \epsilon^d - C\frac{\epsilon^d}{K}. \tag{2.23}$$

Define

$$\Omega_{K+1} = \bigcup_{s=1}^{S} \widetilde{Q}_s^{(K)}.$$

We have $L_j^{(K+1)} = L_j^{(K)} \setminus \pi_j(\Omega_{K+1})$ and

$$\mu\left(L_j^{(K)}\right) \to 0,$$

as $K \to \infty$. This is so because $L_j^{(K)} \setminus \pi_j(\Omega_{K+1})$ consists of the sets $R^{(j,K)}$, $j = 0, \ldots, n$, which have total measure $\leq \frac{n+1}{K}$ plus a set of measure $C\frac{\epsilon^d}{K}$ for each $s = 1, \ldots, S$, which amounts to no more than $\frac{C}{K}$ of measure, as clearly $\epsilon^d S \leq 1$. ∎

Open Problem 5. *Can two lattices in generic position have a* bounded *measurable common tile?*

2.4.2 Multi-lattice tiles: an application to Weyl-Heisenberg bases

Definition 2.3. (Gabor or Weyl-Heisenberg bases)
A Gabor *(or* Weyl-Heisenberg*) basis of $\mathbb{R}^d$ is a function $g \in L^2(\mathbb{R}^d)$, together with two lattices $K = A\mathbb{Z}^d$ (the* translation lattice*) and $L = B\mathbb{Z}^d$ (the* modulation lattice*) such that the collection*

$$\left\{ g(x-\kappa)e^{-2\pi i \lambda x}, \quad \kappa \in K, \ \lambda \in L \right\}, \tag{2.24}$$

is an orthonormal basis of $L^2(\mathbb{R}^d)$.

It had been known for some time (see the introduction and references in [**HW01**]) that if there is a Weyl-Heisenberg basis for the lattices K and L then it must be true that

$$\operatorname{dens} K \cdot \operatorname{dens} L = 1. \tag{2.25}$$

Apart from dimension 1 though, the converse had not been known until Han and Wang [**HW01**] used the idea of multi-lattice tiles to prove that whenever (2.25) holds then there is a g such that collection (2.24) is an orthonormal basis of $L^2(\mathbb{R}^d)$.

Han and Wang [**HW01**] first proved that the genericity condition described in Theorem 2.13 is not necessary when the number of lattices is two.

Theorem 2.14. (Han and Wang, 2001)
Whenever the lattices Λ_0 and Λ_1 in $\mathbb{R}^d$ have the same volume then there exists a measuarable set $E \subseteq \mathbb{R}^d$ which tiles with both of them

Thus, for two lattices of the same volume there is *always* a measurable common tile. This is *not* true for three or more lattices without some condition, as the following result [**K97**] shows.

Theorem 2.15. (Kolountzakis, 1997)
There are three lattices in $\mathbb{R}^2$ which have the same volume and do not admit a common tile.

Proof. Let

$$\Lambda_0 = (2\mathbb{Z}) \times \mathbb{Z},\ \Lambda_1 = \mathbb{Z} \times (2\mathbb{Z}),\ \text{and}\ \Lambda_2 = \left\{(k,l) \in \mathbb{Z}^2 :\ k = l \bmod 2\right\}.$$

It is easy to see that

$$\mathbb{Z}^2 = \sum_{i=0}^{2} \Lambda_i = \bigcup_{i=0}^{2} \Lambda_i.$$

Suppose now that $\Omega \subset \mathbb{R}^2$ is such that for all $x \in \mathbb{R}^2$, outside a set E of measure 0, we have that $x + \Lambda_i$ contains exactly one point of Ω, for all $i = 0, 1, 2$. (We do not assume that Ω is measurable.) It follows that for almost all $x \in \mathbb{R}^2$ (with an exceptional set perhaps different from E) we have

$$\left|(x + \mathbb{Z}^2) \cap \Omega\right| = 2 \text{ and } |(x + \Lambda_i) \cap \Omega| = 1,\ i = 0, 1, 2.$$

Indeed, $\mathbb{Z}^2$ is the disjoint union of Λ_0 and $\Lambda_0 + (1,0)$ and so are all its translates. We define the set

$$E' = E \cup (E - (1,0)),$$

which is clearly still a null set. Then, for $x \notin E'$ the set $x + \mathbb{Z}^2$ contains exactly two points of Ω, since the two disjoint copies of Λ_0 therein both contain exactly one Ω-point.

By translating Ω we may assume that this holds for $x = 0$. Let then

$$\{z, w\} = \mathbb{Z}^2 \cap \Omega.$$

It follows that $z - w \in \mathbb{Z}^2$ and, since $\mathbb{Z}^2 = \bigcup_{j=1}^{3} \Lambda_j$, $z - w$ belongs to some Λ_j. But then the Ω-points z and w belong to the same Λ_j-coset, a contradiction. Hence the Λ_i have no common tile in $\mathbb{R}^2$ in a strong sense.
∎

We continue now with proof of Han and Wang [**HW01**] that (2.25) suffices for the existence of a fucntion g such that the collection (2.24) is a Weyl-Heisenberg basis. Suppose then that (2.25) holds. It follows that the lattices K and L^* have the same volume. Hence, by Theorem 2.14, there is a common tile $E \subseteq \mathbb{R}^d$ for K and L^*. Let

$$g = \chi_E.$$

For any $f \in L^2(\mathbb{R}^d)$ write then

$$f(x) = \sum_{\kappa \in K} f_\kappa(x) := \sum_{\kappa \in K} g(x - \kappa) f(x)$$

which is an orthogonal decomposition precisely because E is a K-tile. For each κ, $f_\kappa(x)$ is a function on $E + \kappa$ which is a L^*-tile. But if a set Ω tiles with a lattice L^* then the collection

$$\left\{\exp^{2\pi i \langle \lambda, x\rangle} :\ \lambda \in L\right\}$$

is an orthogonal basis for $L^2(\Omega)$ (this is merely multi-dimensional Fourier Series plus a change of variable, but see also Theorem 3.2 below). For $\Omega = E + \kappa$ we therefore obtain that

$$f_\kappa(x) = \sum_{\lambda \in L} \langle f_\kappa, e^{2\pi i \langle \lambda, x\rangle}\rangle e^{2\pi i \lambda x} \quad (x \in E + \kappa)$$

is an orthogonal decomposition and so is then

$$f(x) = \sum_{\kappa \in K, \lambda \in L} \langle f, g(x - \kappa) e^{2\pi i \langle \lambda, x\rangle}\rangle g(x - \kappa) e^{2\pi i \lambda x},$$

as as we had to show.

2.5 The support of “soft” multi-lattice tiles

Fix the dimension d and take *any* finite collection of lattices $\Lambda_1, \ldots, \Lambda_N$. Then the function

$$f = \chi_{D_1} * \cdots * \chi_{D_N}, \tag{2.26}$$

where D_j is any tile for Λ_j, tiles with the given lattices, as one can see directly from the definition of tiling (if $f + \Lambda$ is a tiling then so is $f * g + \Lambda$, even for non-lattice Λ).

For this particular f (and whatever choice of D_j) we have

$$\operatorname{diam}\operatorname{supp} f \ge CN,$$

with a constant that dependes only on d. This is easy to see as at least $1/d$ of the sets D_j will be “long” along the same one of the d coordinate axes and the convolution of all of them will therefore also be long along that axis.

If one chooses appropriate parallelograms for the D_j’s one gets more or less the best known (to me at least) construction as regards the diameter of the common tile of the collection $\Lambda_1, \ldots, \Lambda_N$, where, now, we do not insist that the tile be an indicator function, but rather any integrable function. One can in this manner get a tile whose support has diameter $\sim N$.

It is not obvious at all that this size has to grow as a function of N. In fact, the following theorem [**KW99**], which provides a lower bound for the diameter of the support of a common tile, is the only one of its kind, uses (multivariable) entire function theory (some times ineffective in such matters) and is still far from the best known upper bound ($\sim N$).

Theorem 2.16. (Kolountzakis and Wolff, 1997)
Suppose that $\Lambda_1, \ldots, \Lambda_N$ are unimodular lattices in $\mathbb{R}^d$ with $\Lambda_i \cap \Lambda_j = \{0\}$ for all $i \neq j$. Suppose also that the non-zero $f \in L^1(\mathbb{R}^d)$ is a common tile for the Λ_j. Then

$$\operatorname{diam}\operatorname{supp} f \ge C_d N^{1/d}.$$

Proof. All constants below may depend only on the dimension d. We note that $\Lambda_1 \cap \Lambda_2 = \{0\}$ implies that the lattice Λ_1^* is uniformly distributed mod Λ_2^*. This can be proved using Weyl’s lemma–see for example [**K97**].

We shall make use of a theorem of Ronkin [**Ron72**] and Berndtsson [**Ber78**] which concerns the zero set on the real plane of an entire function of several complex variables which is of exponential type. We formulate it as a lemma:

Lemma 2.3. (Ronkin 1972, Berndtsson 1978)
Assume that $E \subset \mathbb{R}^d$ is a countable set with any two points having distance at least h and let

$$d_E = \limsup_{r\to\infty} \frac{|E \cap D(0,r)|}{|D(0,r)|}$$

be its upper density (see Definition 1.2). Assume that $g : \mathbb{C}^d \to \mathbb{C}$ is an entire function vanishing on E which is of exponential type

$$\sigma < A(d) h^{d-1} d_E.$$

Then g is identically 0. (Here $A(d)$ is an explicit function of the dimension d.)

When $d = 1$ this is classical and follows from Jensen’s formula.

Assume that $f : \mathbb{R}^d \to \mathbb{C}$ is as in Theorem 2.16 and write

$$\alpha = \operatorname{diam}\ \operatorname{supp} f$$

We may assume that supp$f$ is contained in a disc of radius $\lesssim \alpha$ centered at the origin, since the assumptions are unaffected by a translation of coordinates. Then $\widehat{f}$ can be extended to $\mathbb{C}^d$ as an entire function of exponential type $C\alpha$, in fact

$$\left|\widehat{f}(x+iy)\right| \le C_f e^{C\alpha|y|}, \quad \text{for } x+iy \in \mathbb{C}^d.$$

Furthermore, since f tiles with all Λ_j, it follows that $\widehat{f}$ vanishes on

$$\mathcal{Z} = \bigcup_{i=1}^{n} \Lambda_i^* \setminus \{0\}.$$

Observe that, since every lattice Λ_i^* is uniformly distributed mod every Λ_j^*, $j \ne i$, the density of points in each Λ_i^* which are also in some Λ_j^* is 0 and therefore the density of the set $\mathcal{Z}$ is equal to n.

In order to use Lemma 2.3 we have to select a large (in terms of upper density), well-separated subset of $\mathcal{Z}$. Notice first that we can assume that for each i all points of Λ_i^* are at least distance $n^{-\frac{1}{d}}$ apart. For if $u, v \in \Lambda_i^*$ have $|u-v| < n^{-\frac{1}{d}}$ then, for a suitable constant c, the one-dimensional version of Lemma 2.3 implies that the function $\widehat{f}$ on the subspace $E = \mathbb{C}(u-v)$ cannot be of exponential type $\le cn^{\frac{1}{d}}$. Indeed, $\widehat{f}$ would have too many zeros on that subspace, namely all multiples of $u-v$, which all belong to Λ_i^*. Note also that $\widehat{f}$ does not vanish identically on this subspace. But $\widehat{f}$ restricted to E is the Fourier Transform of $f_E : E \to \mathbb{C}$ defined by $f_E(x) = \int_{x+E^\perp} f(y)\,dy$ (here $E^\perp$ is the orthogonal complement of $E \cap \mathbb{R}^n$ in $\mathbb{R}^n$). Hence $\alpha \ge \operatorname{diam} \operatorname{supp} f_E \ge Cn^{\frac{1}{d}}$, which is what we want to conclude about α.

Suppose now that we want to extract a subset of $\mathcal{Z}$ whose elements are at least h distance apart, for some $h > 0$ to be determined later. We shall say that point x of lattice Λ_i^* is *killed* by point y of lattice Λ_j^* if $|x-y| < h$. Then, we define the subset $\mathcal{Z}'$ of $\mathcal{Z}$ as those points of $\mathcal{Z}$ which are not killed by any point of the *other* lattices. This set clearly has all its points at distance at least h apart, provided that

$$h \le \frac{1}{2} \min_{u,v \in \Lambda_i^*} |u-v| \le Cn^{-\frac{1}{d}}, \tag{2.27}$$

so that no point of a lattice may kill a point of the same lattice. Let us see how many points of Λ_2^* are killed by some point of Λ_1^*. We use the uniform distribution of Λ_2^* mod Λ_1^*.

Fix a fundamental parallelepiped D_1 of Λ_1^*. It is clear that only a fraction $\rho(h) \le Ch^d$ of $D_1 = \mathbb{R}^d/\Lambda_1^*$ has distance from 0 that is less than h (this distance is measured on the *torus* D_1). As Λ_2^* is uniformly distributed mod Λ_1^* the subset of points of Λ_2^* which are killed by some point of Λ_1^* has density $\rho(h)$. Hence the density of those points of Λ_2^* that are killed by *any* other lattice is at most $(n-1)\rho(h) \le Ch^d n$. We deduce that the density of $\mathcal{Z}'$ is at least $(1 - Cnh^d)n$. We now choose $h = cn^{-\frac{1}{d}}$, for a sufficiently small constant c, to ensure that the density of $\mathcal{Z}'$ is at least Cn. Applying Lemma 2.3 with $g = \widehat{f}$ and $E = \mathcal{Z}'$ we get

$$\alpha \ge CAh^{d-1}n \ge Cn^{\frac{1}{d}}.$$

∎

Open Problem 6. *Bridge the gap between Theorem 2.16 and the upper bound $\sim N$.*

3 Lecture 3: The Fuglede Conjecture

3.1 Spectral sets and tiling

Let us write $e_\lambda(x) = \exp 2\pi i\langle\lambda, x\rangle$.

Definition 3.1. (Spectral sets)
Suppose that Ω is a bounded open set of measure 1. We call Ω spectral *if $L^2(\Omega)$ has an orthonormal basis*

$$E_\Lambda = \{e_\lambda : \ \lambda \in \Lambda\}$$

of exponentials. The set Λ is then called a spectrum *for Ω.*

(We only restrict ourselves to sets of measure 1 to make our life simpler.)

The inner product and norm on $L^2(\Omega)$ are

$$\langle f, g\rangle_\Omega = \int_\Omega f\overline{g}, \quad \text{and} \quad \|f\|_\Omega^2 = \int_\Omega |f|^2.$$

We have

$$\langle e_\lambda, e_x\rangle_\Omega = \widehat{\chi_\Omega}(x-\lambda).$$

which gives

$$E(\Lambda) \text{ is orthogonal} \Leftrightarrow \forall \lambda, \mu \in \Lambda, \lambda \neq \mu : \ \widehat{\chi_\Omega}(\lambda - \mu) = 0$$

For $E(\Lambda)$ to be complete as well we must in addition have

$$\forall f \in L^2(\Omega) : \quad \|f\|_2^2 = \sum_{\lambda\in\Lambda} |\langle f, e_\lambda\rangle|^2. \tag{3.1}$$

It is sufficient to have (3.1) for $f(t) = e_x(t)$, $x \in \mathbb{R}^d$, since then we have it in the closed linear span of these functions, which is all of $L^2(\Omega)$.

An equivalent reformulation for Λ to be a spectrum of Ω is therefore the following, which we state as a theorem.

Theorem 3.1. *The set Λ is a spectrum of Ω if and only if $\sum_{\lambda\in\Lambda} \left|\widehat{\chi_\Omega}(x-\lambda)\right|^2 = 1$, for almost every $x \in \mathbb{R}^d$. In tiling language*

$$\Lambda \text{ is a spectrum of } \Omega \ \Leftrightarrow \ \left|\widehat{\chi_\Omega}\right|^2 + \Lambda = \mathbb{R}^d$$

The relevant functions are shown in Figure 11, for the case of Ω being an interval.

It follows from Theorem 3.1 that the spectrum Λ of domain Ω, if it exists, has all the nice properties of tiling sets. In particular, Λ has uniform density equal to 1 and its points are ϵ-separated for some $\epsilon > 0$.

We can now state Fuglede's Conjecture [**Fug74**]

Conjecture 3.1. (Fuglede 1974)
Let $\Omega \subseteq \mathbb{R}^d$ be a bounded, open domain of measure 1. Then Ω is spectral if and only if it can tile space by translation.

We should emphasize here that no relation is claimed in the conjecture between the spectrum of Ω and the set of translations with which Ω tiles.

Remark 3.1. *By the preceding discussion Fuglede's Conjecture states that Ω is a tile if and only if $\left|\widehat{\chi_\Omega}\right|^2$ is a tile (both tilings are at level 1).*

Despite a lot of work that has been done in the last 5-6 years the conjecture remains open in all dimensions and in both directions. One easy and important case though is given by the following [**Fug74**].

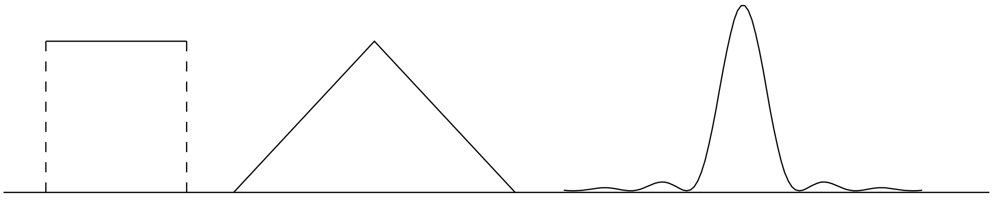


FIGURE 11: The functions χ_Ω, $\chi_\Omega * \widetilde{\chi_\Omega}$ and $|\widehat{\chi_\Omega}|^2$, when Ω is an interval. The last two functions are a Fourier Transform pair.

Theorem 3.2. (Fuglede, 1974)
Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded open domain of measure 1 and $\Lambda \subseteq \mathbb{R}^d$ a lattice of density 1. Then $\Omega + \Lambda = \mathbb{R}^d$ if and only if Λ^* (the dual lattice) is a spectrum of Ω.

Proof. As remarked above, Λ^* is a spectrum of Ω if and only if (see §1.2.1)

$$|\widehat{\chi_\Omega}|^2 + \Lambda^* = \mathbb{R}^d,$$

which is in turn equivalent to the Fourier Transform of the function $|\widehat{\chi_\Omega}|^2$ vanishing on the dual lattice of Λ^* except at 0. That is the function $f = \chi_\Omega * \widetilde{\chi_\Omega}$ vanishes on $\Lambda \setminus \{0\}$. But f is non-zero exactly on $\Omega - \Omega$, hence the above vanishing is equivalent to

$$(\Omega - \Omega) \cap \Lambda = \{0\}$$

which means precisely that the copies $\Omega + \lambda$, $\lambda \in \Lambda$, are non-overlapping. But $|\Omega| = 1$ and $\operatorname{dens} \Lambda = 1$, hence the above packing is indeed a tiling. The argument is completely reversible. ∎

3.2 Implications of the Brunn-Minkowski inequality for convex tiles and spectral bodies

Let us recall a simple case of the **Brunn-Minkowski Inequality** (see e.g. [**S93**]).

For a convex body K we always have

$$|K - K| \geq 2^d |K|.$$

We have equality above exactly when K is symmetric, in which case $K - K = 2K$.

Using the Brunn-Minkowski inequality one can show:

Theorem 3.3. (**Minkowski, ca. 1900**)
If K is a convex translational tile then it is symmetric.

Proof. Suppose K is convex and $K + \Lambda = \mathbb{R}^d$. By the packing condition (non-overlaping of translates) only we get

$$(K - K) \cap (\Lambda - \Lambda) = \{0\}.$$

Define the convex set $L = \frac{1}{2}(K - K)$. One easily sees that $L - L = K - K$, so that

$$L + \Lambda \leq \mathbb{R}^d,$$

is a packing. But this implies (see Lemma 1.3)

$$|L| \le 1,$$

and by the equality case in the Brunn-Minkowski inequality K is symmetric. ∎

The following theorem [**K00**] is also a consequence of the Brunn-Minkowski inequality.

Theorem 3.4. (Kolountzakis, 2000)
If Ω is convex and spectral then it is symmetric.

This result is of course in agreement with the Fuglede Conjecture as this would be false if there were any non-symmetric convex spectral domains. We prove Theorem 3.4 in §3.2.1 and §3.2.2 below.

3.2.1 Fourier-analytic conditions for tiling

When studying tiling by the function $\left|\widehat{\chi_\Omega}\right|^2$ Theorem 1.1 is not applicable since the Fourier Transform of the function, namely $\chi_\Omega * \widetilde{\chi_\Omega}$, is never smooth. However, the positivity of the function and its Fourier Transform as well as the compact support of the Fourier Transform compensate for this lack of smoothness and allow us to prove the following result [**K00**].

Theorem 3.5. (Kolountzakis, 2000)
Suppose that $f \ge 0$ is not identically 0, that $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \ge 0$ has compact support and $\Lambda \subset \mathbb{R}^d$. If $f + \Lambda$ is a tiling then

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \left\{ x \in \mathbb{R}^d : \ \widehat{f}(x) = 0 \right\} \cup \{0\}. \tag{3.2}$$

Proof. Assume that $f + \Lambda = w\mathbb{R}^d$ and let

$$K = \left\{ \widehat{f} = 0 \right\} \cup \{0\}.$$

We have to show that

$$\widehat{\delta_\Lambda}(\phi) = 0, \quad \forall \phi \in C_c^\infty(K^c).$$

Since $\widehat{\delta_\Lambda}(\phi) = \delta_\Lambda(\widehat{\phi})$ this is equivalent to $\sum_{\lambda\in\Lambda} \widehat{\phi}(\lambda) = 0$, for each such ϕ. Notice that $h = \phi/\widehat{f}$ is a continuous function, but not necessarily smooth. We shall need that $\widehat{h} \in L^1$. This is a consequence of a well-known theorem of Wiener [**R73**, Ch. 11]. We denote by $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional torus.

Theorem 3.6. (Wiener)
If $g \in C(\mathbb{T}^d)$ has an absolutely convergent Fourier series

$$g(x) = \sum_{n\in\mathbb{Z}^d} \widehat{g}(n) e^{2\pi i \langle n, x\rangle}, \quad \widehat{g} \in \ell^1(\mathbb{Z}^d),$$

and if g does not vanish anywhere on $\mathbb{T}^d$ then $1/g$ also has an absolutely convergent Fourier series.

Assume that

$$\operatorname{supp}\phi,\ \operatorname{supp}\widehat{f} \subseteq \left(-\frac{L}{2}, \frac{L}{2}\right)^d.$$

Define the function F to be:
(i) periodic in $\mathbb{R}^d$ with period lattice $(L\mathbb{Z})^d$,
(ii) to agree with $\widehat{f}$ on $\operatorname{supp}\phi$,
(iii) to be non-zero everywhere and,
(iv) to have $\widehat{F} \in \ell^1(\mathbb{Z}^d)$, i.e.,

$$\widehat{F} = \sum_{n\in\mathbb{Z}^d} \widehat{F}(n) \delta_{L^{-1}n},$$

is a finite measure in $\mathbb{R}^d$.

One way to define such an F is as follows. First, define the $(L\mathbb{Z})^d$-periodic function $g \ge 0$ to be $\widehat{f}$ periodically extended. The Fourier coefficients of g are $\widehat{g}(n) = L^{-d} f(-n/L) \ge 0$. Since $g, \widehat{g} \ge 0$ and g is continuous at 0 it is easy to prove that $\sum_{n\in\mathbb{Z}^d} \widehat{g}(n) = g(0)$, and therefore that g has an absolutely convergent Fourier series.

Let ϵ be small enough to guarantee that $\widehat{f}$ (and hence g) does not vanish on $(\operatorname{supp}\phi) + B_\epsilon(0)$. Let k be a smooth $(L\mathbb{Z})^d$-periodic function which is equal to 1 on $(\operatorname{supp}\phi) + (L\mathbb{Z}^d)$ and equal to 0 off $(\operatorname{supp}\phi + B_\epsilon(0)) + (L\mathbb{Z}^d)$, and satisfies $0 \le k \le 1$ everywhere. Finally, define

$$F = kg + (1-k).$$

Since both k and g have absolutely summable Fourier series and this property is preserved under both sums and products, it follows that F also has an absolutely summable Fourier series. And by the nonnegativity of g we get that F is never 0, since $k = 0$ on $\left\{\widehat{f} = 0\right\} + (L\mathbb{Z}^d)$.

By Wiener's Theorem 3.6, $\widehat{F^{-1}} \in \ell^1(\mathbb{Z}^d)$, i.e., $\widehat{F^{-1}}$ is a finite measure on $\mathbb{R}^d$. We now have that

$$\left(\frac{\phi}{\widehat{f}}\right)^\wedge = \widehat{\phi F^{-1}} = \widehat{\phi} * \widehat{F^{-1}} \in L^1(\mathbb{R}^d).$$

This justifies the interchange of the summation and integration below:

$$\begin{aligned}
\sum_{\lambda\in\Lambda} \widehat{\phi}(\lambda) &= \sum_{\lambda\in\Lambda} \left(\frac{\phi}{\widehat{f}}\widehat{f}\right)^\wedge (\lambda) \\
&= \sum_{\lambda\in\Lambda} \left(\frac{\phi}{\widehat{f}}\right)^\wedge * \widehat{\widehat{f}}\,(\lambda) \\
&= \sum_{\lambda\in\Lambda} \int_{\mathbb{R}^d} \left(\frac{\phi}{\widehat{f}}\right)^\wedge (y) f(y-\lambda)\, dy \\
&= \int_{\mathbb{R}^d} \left(\frac{\phi}{\widehat{f}}\right)^\wedge (y) \sum_{\lambda\in\Lambda} f(y-\lambda)\, dy \\
&= w \int_{\mathbb{R}^d} \left(\frac{\phi}{\widehat{f}}\right)^\wedge (y)\, dy \\
&= w \frac{\phi}{\widehat{f}}(0) \\
&= 0,
\end{aligned}$$

as we had to show. ∎

For a set $A \subseteq \mathbb{R}^d$ and $\delta > 0$ we write

$$A_\delta = \left\{x \in \mathbb{R}^d :\ \operatorname{dist}(x, A) < \delta\right\}.$$

We shall need the following partial converse to Theorem 3.5 (see Figure 12 for the assumptions of Theorem 3.7).

Theorem 3.7. *Suppose that $f \in L^1(\mathbb{R}^d)$, and that $\Lambda \subset \mathbb{R}^d$ has uniformly bounded density. Suppose also that $O \subset \mathbb{R}^d$ is open and*

$$\operatorname{supp}\widehat{\delta_\Lambda} \setminus \{0\} \ \subseteq\ O \quad \textit{and} \quad O_\delta \subseteq \left\{\widehat{f} = 0\right\}, \tag{3.3}$$

for some $\delta > 0$. Then $f + \Lambda$ is a tiling at level $\widehat{f}(0) \cdot \widehat{\delta_\Lambda}(\{0\})$.

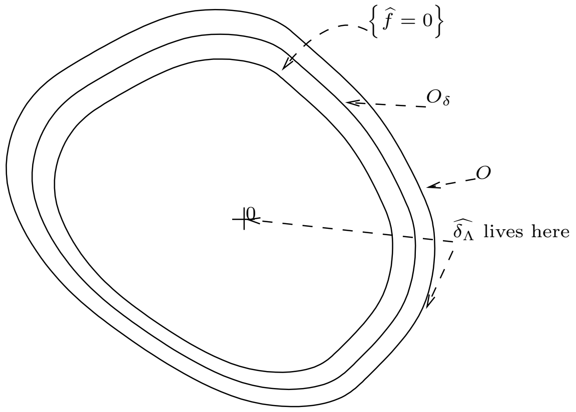


FIGURE 12: The sets appearing in Theorem 3.7. The sets $O, O_\delta, \left\{\widehat{f}=0\right\}$ all live *outside* the contours.

The assumptions of Theorem 3.7 ensure that the supports of $\widehat{\delta_\Lambda}$ (except at 0) and $\widehat{f}$ are well separated. In other words $\widehat{f}$ vanishes to infinite order on the support of $\widehat{\delta_\Lambda}$. This makes the formal implication

$$\widehat{f}\cdot\widehat{\delta_\Lambda}=\ell\delta_0 \Longrightarrow f*\delta_\Lambda=\ell$$

correct.

Remark 3.2. *By the assumptions of the theorem we know that $\widehat{\delta_\Lambda}$ is supported only at* 0, *in a neighborhood of the origin. It follows from Theorem 1.11 that $\widehat{\delta_\Lambda}$ is a measure in some neighborhood of the origin so it makes sense to speak of $\widehat{\delta_\Lambda}(\{0\})$.*

Proof. Let $\psi:\mathbb{R}^d\to\mathbb{R}$ be smooth, have support in $B_1(0)$ and $\widehat{\psi}(0)=1$ and for $\epsilon>0$ define the approximate identity $\psi_\epsilon(x)=\epsilon^{-d}\psi(x/\epsilon)$. Let

$$f_\epsilon=\widehat{\psi_\epsilon}f,$$

which has rapid decay.

First we show that $(\int f_\epsilon)^{-1}f_\epsilon+\Lambda$ is a tiling. That is, we show that the convolution $f_\epsilon*\delta_\Lambda$ is a constant. Let ϕ be any Schwartz function. Then

$$f_\epsilon*\delta_\Lambda(\phi)=\widehat{f_\epsilon}\widehat{\delta_\Lambda}(\widehat{\phi}(-x))=\widehat{\delta_\Lambda}(\widehat{\phi}(-x)\widehat{f_\epsilon}).$$

The function $\widehat{\phi}(-x)\widehat{f_\epsilon}$ is a Schwartz function whose support intersects $\operatorname{supp}\widehat{\delta_\Lambda}$ only at 0, since, for small enough $\epsilon>0$,

$$\operatorname{supp}\widehat{\phi}\widehat{f_\epsilon}\subseteq\operatorname{supp}\widehat{f_\epsilon}\subseteq(\operatorname{supp}\widehat{f})_\epsilon\subseteq O^c.$$

Hence, for each Schwartz function ϕ

$$f_\epsilon*\delta_\Lambda(\phi)=\widehat{\phi}(0)\widehat{f_\epsilon}(0)\widehat{\delta_\Lambda}(\{0\}),$$

which implies

$$f_\epsilon*\delta_\Lambda(x)=\widehat{f_\epsilon}(0)\widehat{\delta_\Lambda}(\{0\}),\quad \text{a.e.}(x).$$

We also have that $\sum_{\lambda\in\Lambda}|f(x-\lambda)|$ is finite a.e. (see the remark following the definition of tiling), hence, for almost every $x\in\mathbb{R}^d$

$$\sum_{\lambda\in\Lambda}|f(x-\lambda)-f_\epsilon(x-\lambda)|=\sum_{\lambda\in\Lambda}|f(x-\lambda)|\cdot\left|1-\widehat{\psi_\epsilon}(x-\lambda)\right|,$$

which tends to 0 as $\epsilon\to 0$. This proves

$$\sum_{\lambda\in\Lambda}f(x-\lambda)=\widehat{f}(0)\cdot\widehat{\delta_\Lambda}(\{0\}),\quad\text{a.e.}(x).$$

∎

3.2.2 Convex spectral bodies must be symmetric

Proof of Theorem 3.4: Write $K = \Omega - \Omega$, which is a symmetric, open convex set. Assume that (Ω, Λ) is a spectral pair. We can clearly assume that $0 \in \Lambda$. It follows that $\left|\widehat{\chi_\Omega}\right|^2 + \Lambda$ is a tiling and hence that Λ has uniformly bounded density, has density equal to 1 and $\widehat{\delta_\Lambda}(\{0\}) = 1$.

By Theorem 3.5 (with $f = \left|\widehat{\chi_\Omega}\right|^2$, $\widehat{f} = \chi_\Omega * \widetilde{\chi_\Omega}(-x)$) it follows that

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \{0\} \cup K^c.$$

Let $H = K/2$ and write

$$f(x) = \chi_H * \widetilde{\chi_H}(x) = \int_{\mathbb{R}^d} \chi_H(y) \chi_H(y-x)\, dy.$$

The function f is supported in $\overline{K}$ and has nonnegative Fourier Transform

$$\widehat{f} = \left|\widehat{\chi_H}\right|^2.$$

We have

$$\int_{\mathbb{R}^d} \widehat{f} = f(0) = \operatorname{vol} H$$

and

$$\widehat{f}(0) = \int_{\mathbb{R}^d} f = (\operatorname{vol} H)^2.$$

By the Brunn-Minkowski inequality for any convex body Ω,

$$\operatorname{vol} \frac{1}{2}(\Omega - \Omega) \geq \operatorname{vol} \Omega,$$

with equality only in the case of symmetric Ω. Since Ω has been assumed to be non-symmetric it follows that

$$\operatorname{vol} H > 1.$$

For

$$1 > \rho > \left(\frac{1}{\operatorname{vol} H}\right)^{1/d}$$

consider

$$g(x) = f(x/\rho)$$

which is supported properly inside K, and has

$$g(0) = f(0) = \operatorname{vol} H, \quad \int_{\mathbb{R}^d} g = \rho^d \int_{\mathbb{R}^d} f = \rho^d (\operatorname{vol} H)^2.$$

Since $\operatorname{supp} g$ is properly contained in K Theorem 3.7 implies that $\widehat{g} + \Lambda$ is a tiling at level $\int \widehat{g} \cdot \operatorname{dens} \Lambda = \int \widehat{g} = g(0) = \operatorname{vol} H$. However, the value of $\widehat{g}$ at 0 is $\int g = \rho^d (\operatorname{vol} H)^2 > \operatorname{vol} H$, and, since $\widehat{g} \geq 0$ and $\widehat{g}$ is continuous, this is a contradiction. ∎

3.3 The spectra of the cube

In this section we prove the following [**IP98**, **LRW00**, **K00b**].

Theorem 3.8. (Iosevich and Pedersen, 1998, Lagarias, Reeds and Wang 1998, Kolountzakis 1999)
Let $Q = (-1/2, 1/2)^d$ be the unit cube in $\mathbb{R}^d$ and $\Lambda \subseteq \mathbb{R}^d$. Then

$$\Lambda \textit{ is a spectrum of } Q \Leftrightarrow Q + \Lambda = \mathbb{R}^d.$$

This had been proved earlier by Jorgensen and Pedersen [**JP99**] for $d = 3$.

3.3.1 A lemma for two different tiles

The following simple result is rather unexpected. It is intuitively clear when Λ is a periodic set but it is, perhaps, suprising that it holds without any assumptions on the set Λ.

Lemma 3.1. *If $f, g \ge 0$, $\int f(x)dx = \int g(x)dx = 1$ and both $f+\Lambda$ and $g+\Lambda$ are packings of $\mathbb{R}^d$, then $f+\Lambda$ is a tiling if and only if $g+\Lambda$ is a tiling.*

Proof. We first show that, under the assumptions of the Theorem,

$$f+\Lambda \text{ tiles } -\operatorname{supp} g \implies g+\Lambda \text{ tiles } -\operatorname{supp} f. \tag{3.4}$$

Indeed, if $f+\Lambda$ tiles $-\operatorname{supp} g$ then

$$1 = \int g(-x) \sum_{\lambda\in\Lambda} f(x-\lambda)\ dx = \sum_{\lambda\in\Lambda} \int g(-x) f(x-\lambda)\ dx,$$

which, after the change of variable $y = -x+\lambda$, gives

$$1 = \int f(-y) \sum_{\lambda\in\Lambda} g(y-\lambda)\ dy.$$

This in turn implies, since $\sum_{\lambda\in\Lambda} g(y-\lambda) \le 1$, that $\sum_\lambda g(y-\lambda) = 1$ for a.e. $y \in -\operatorname{supp} f$.

To complete the proof of the theorem, notice that if $f+\Lambda$ is a tiling of $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ is arbitrary then both $f(x-a)+\Lambda$ and $g(x-a)+\Lambda$ are packings and $f+\Lambda$ tiles $-\operatorname{supp} g(x-a) = -\operatorname{supp} g - a$. We conclude that $g(x-a)+\Lambda$ tiles $-\operatorname{supp} f$, or $g+\Lambda$ tiles $-\operatorname{supp} f - a$. Since $a\in\mathbb{R}^d$ is arbitrary we conclude that $g+\Lambda$ tiles $\mathbb{R}^d$. ∎

Example: Use Lemma 3.1 to prove that there is no measurable nonnegative function f that tiles with $\Lambda = \mathbb{Z}^d \setminus \{0\}$ (or even $\mathbb{Z}^d$ minus a set of lower density 0, such as a line). Try to prove this otherwise.

3.3.2 Failure of the lemma for non-translational tiling

Suppose we study tiling where all rigid motions of the tile, and not just translations, are allowed. The analogue of the tiling set then is a set Λ of rigid motions. For $x \in \mathbb{R}^d$ and λ a rigid motion we denote by $\lambda(x)$ the action of λ on x. The following theorem shows that our Lemma 3.1 is very particular to translations.

Theorem 3.9. *There are two polygons A and B in $\mathbb{R}^2$ of the same area and a set of rigid motions Λ such that both collections $\{\lambda(A) :\ \lambda\in\Lambda\}$ and $\{\lambda(B) :\ \lambda\in\Lambda\}$ are packing but only one of them is a tiling.*

Proof. Take $A = (-1/2, 1/2)^2$ and B to be the parallelogram with vertices $(-1/2,-1/2)$, $(1/2,0)$, $(1/2,1)$ and $(-1/2,1/2)$. Take the set of rigid motions to be the set of translations by $\mathbb{Z}^2$ modified as follows: instead of translating by the elements $(0,k)$, $k<0$, we first reflect the domain with respect to the x-axis and then translate it by $(0,k)$. For the elements (m,n) of $\mathbb{Z}^2$ where either $m\ne 0$ or $n \ge 0$ we just translate.

Since the reflection has no effect on A the collection $\{\lambda(A) :\ \lambda\in\Lambda\}$ clearly constitutes a tiling. On the other hand the collection $\{\lambda(B) :\ \lambda\in\Lambda\}$ can be seen in Figure 13 and is clearly not a tiling, although it is a packing. ∎

3.3.3 Deducing tiling from the condition on supports

Assume that we have

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \left\{\widehat{f}=0\right\} \cup \{0\} \tag{3.5}$$

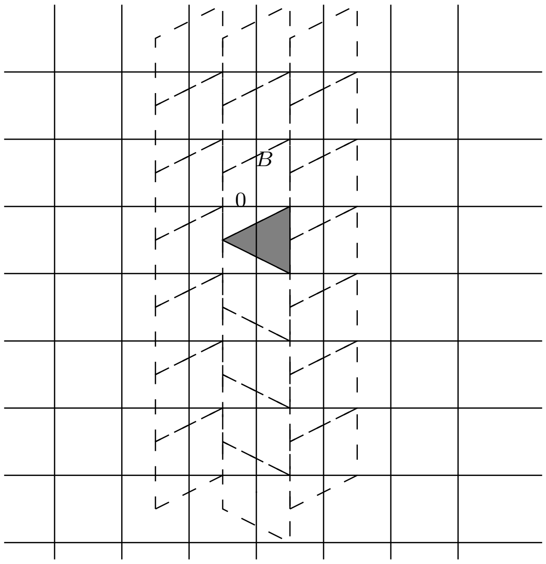


FIGURE 13: Packing of set B, the parallelogram above the shaded triangle, with motions Λ. The shaded triangle is not covered.

for some non-zero $f \ge 0$ in L^1 and that Λ is of bounded density. Since $\widehat{f}(0) = \int f > 0$ it follows that in some neighborhood N of 0 we have $(\operatorname{supp} \widehat{\delta_\Lambda}) \cap N = \{0\}$. Hence the set

$$O = \left(\operatorname{supp} \widehat{\delta_\Lambda} \setminus \{0\}\right)^c \tag{3.6}$$

is open and

$$\left\{\widehat{f} \neq 0\right\} \subseteq O.$$

We shall need the following result.

Theorem 3.10. *Suppose that* $0 \le f \in L^1(\mathbb{R}^d)$, $\int f = 1$, Λ *(of uniformly bounded density) is of density* 1, *and that* (3.5) *holds. Suppose also that for the open set* O *of* (3.6) *and for each* $\epsilon > 0$ *there exists* $f_\epsilon \ge 0$ *in* $L^1(\mathbb{R}^d)$ *such that* $\widehat{f_\epsilon}$ *is in* C^∞, $\operatorname{supp} \widehat{f_\epsilon} \subseteq O$ *and*

$$\|f - f_\epsilon\|_1 \le \epsilon.$$

Then $f + \Lambda$ *is a tiling.*

Proof. Suppose that f_ϵ is as in the Theorem. First we show that $(\int f_\epsilon)^{-1} f_\epsilon + \Lambda$ is a tiling. That is, we show that the convolution $f_\epsilon * \delta_\Lambda$ is a constant. Let ϕ be C_c^∞ function. Then

$$(f_\epsilon * \delta_\Lambda)(\phi) = \widehat{f_\epsilon}\widehat{\delta_\Lambda}(\widehat{\phi}) = \widehat{\delta_\Lambda}(\widehat{\phi}\widehat{f_\epsilon}).$$

But the function $\widehat{\psi} = \widehat{\phi}\widehat{f_\epsilon}$ is a C_c^∞ function whose support intersects $\operatorname{supp} \widehat{\delta_\Lambda}$ only at 0. And, it is not hard to show, because Λ has density 1, that $\widehat{\delta_\Lambda}$ is equal to δ_0 in a neighborhood of 0 (see [**K00a**]). Hence

$$(f_\epsilon * \delta_\Lambda)(\phi) = \left(\widehat{\phi}\widehat{f_\epsilon}\right)(0) = \int \phi \int f_\epsilon,$$

and, since this is true for an arbitrary C_c^∞ function ϕ, we conclude that $f_\epsilon * \delta_\Lambda = \int f_\epsilon$, as we had to show.

For any set Λ of uniformly bounded density we have (B is any ball in $\mathbb{R}^d$ and $g \in L^1(\mathbb{R}^d)$)

$$\int_B \left|\sum_{\lambda\in\Lambda} g(x-\lambda)\right| dx \le C_{B,\Lambda} \int_{\mathbb{R}^d} |g|,$$

(See [**KL96**] for a proof of this in dimension 1, which holds for any dimension.) Applying this for $g = f - f_\epsilon$ we obtain that

$$\sum_{\lambda\in\Lambda} f_\epsilon(x-\lambda) \to \sum_{\lambda\in\Lambda} f(x-\lambda), \quad \text{in } L^1(B).$$

Since B is arbitrary this implies that $\sum_{\lambda\in\Lambda} f(x-\lambda) = 1$, a.e. in $\mathbb{R}^d$. ∎

We write $\widetilde{f}(x) = \overline{f(-x)}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of measure 1, χ_Ω its indicator function and f be such that $\widehat{f} = \chi_\Omega * \widetilde{\chi_\Omega}$. Then $\widetilde{f} = |\widehat{\chi_\Omega}|^2 \ge 0$, $\int f = 1$ by Parseval's theorem. Clearly we have $\left\{\widehat{f} \ne 0\right\} = \Omega - \Omega$.

Write

$$\Omega_\epsilon = \{x \in \Omega :\ \mathrm{dist}\,(x, \partial\Omega) > \epsilon\},$$

and define f_ϵ by

$$\widehat{f_\epsilon} = \psi_\epsilon * \chi_{\Omega_\epsilon} * (\psi_\epsilon * \chi_{\Omega_\epsilon})^{\widetilde{}}$$

(or $\widetilde{f_\epsilon} = \left|\widehat{\psi_\epsilon}\right|^2 |\widehat{\chi_{\Omega_\epsilon}}|^2$), where ψ_ϵ is a smooth, positive-definite approximate identity supported in $B_{\epsilon/2}(0)$.

One can easily prove the following proposition.

If $g_n \to g$ in L^2 then $|g_n|^2 \to |g|^2$ in L^1.

(For the proof just notice the identity

$$|g|^2 - |g_n|^2 = |g - g_n|^2 + 2 \cdot \mathrm{Re}\left(\overline{g_n}(g - g_n)\right),$$

integrate and use the triangle and Cauchy-Schwartz inequalities.)

Since $\psi_\epsilon * \chi_{\Omega_\epsilon} \to \chi_\Omega$ in L^2 (dominated convergence) we have (Parseval) that $\widehat{\psi_\epsilon}\widehat{\chi_{\Omega_\epsilon}} \to \widehat{\chi_\Omega}$ in L^2 and, using the proposition above, that $\left|\widehat{\psi_\epsilon}\right|^2 |\widehat{\chi_{\Omega_\epsilon}}|^2 \to |\widehat{\chi_\Omega}|^2$ in L^1, which means that $f_\epsilon \to f$ in L^1.

We also have that

$$\mathrm{supp}\,\widehat{f_\epsilon} \subseteq \overline{\Omega_{\epsilon/2}} - \overline{\Omega_{\epsilon/2}} \subseteq \Omega - \Omega = \left\{\widehat{f} \ne 0\right\}.$$

The assumptions of Theorem 3.10 are therefore satisfied. Combining Theorems 3.5 and 3.10 with the above observations we obtain the following characterization of tiling by the function $|\widehat{\chi_\Omega}|^2$. The special form of this function allows us to drop any conditions, that are otherwise needed, regarding the order (how many derivatives it involves) of the tempered distribution $\widehat{\delta_\Lambda}$.

Theorem 3.11. *Let Ω be a bounded open set, Λ a discrete set in $\mathbb{R}^d$, and $\delta_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$. Then $|\widehat{\chi_\Omega}|^2 + \Lambda$ is a tiling if and only if Λ has uniformly bounded density and*

$$(\Omega - \Omega) \cap \mathrm{supp}\,\widehat{\delta_\Lambda} = \{0\}.$$

Proof of Theorem 3.8. By a simple calculation we get

$$\begin{aligned} \mathcal{Z}(\widehat{\chi_Q}) &= \left\{\xi \in \mathbb{R}^d :\ \text{some } \xi_j \text{ is a non-zero integer}\right\} \\ &\subseteq (2Q)^c. \end{aligned}$$

Suppose first that $Q + \Lambda = \mathbb{R}^d$. From Theorem 3.5 it follows that

$$\begin{aligned} \mathrm{supp}\,\widehat{\delta_\Lambda} &\subseteq \{0\} \cup \mathcal{Z}(\widehat{\chi_Q}) \\ &\subset \{0\} \cup (Q - Q)^c \end{aligned}$$

and from Theorem 3.11 we deduce that Λ is a spectrum of Q.

Conversely assume that Λ is a spectrum of Q, so that $|\widehat{\chi_Q}|^2+\Lambda=\mathbb{R}^d$. It follows that $(Q-Q)\cap(\Lambda-\Lambda)=\{0\}$ as we have $|\widehat{\chi_Q}|^2(0)=1$ and $|\widehat{\chi_Q}|^2>0$ on $Q-Q$. But this means that we have a packing $Q+\Lambda\le\mathbb{R}^d$. However, Λ is a tiling set, because it is a spectrum, and there is another object that tiles with Λ, namely $|\widehat{\chi_Q}|^2$, and this object has the same integral as χ_Q (that is, 1). It follows from Lemma 3.1 that $Q+\Lambda=\mathbb{R}^d$ is also a tiling, as we had to prove. ∎

3.4 A proof that the disk is not spectral, which just makes it

Here we present a proof of why the disk $D=\left\{|x|<\frac{1}{\sqrt{\pi}}\right\}$ in the plane is not a spectral domain. The radius is taken equal to $1/\sqrt{\pi}$ to make the disk have area 1, as we usually do in this survey.

The proof is simple but relies on two not-so-easy facts.

1. The first is the upper bound $\frac{\pi}{\sqrt{12}}$, due to Thue, on the density of any packing of the plane with copies of the same disk (see, for example, [**PA95**, Ch. 3]).
2. The second is that the first zero of the Fourier Transform of the indicator function of D is at distance approximately 1.08098 from the origin. This may either be looked up in tables of the Bessel function J_1 (which, up to scaling, is the Fourier Transform of the indicator function of D restricted on a line) or may be computed in a straightforward way using a computer. (The Fourier Transform of the unit-area disk, defined by $\widehat{\chi_D}(\xi)=\int_D \exp(-2\pi i\xi x)\,dx$, is equal to a constant times $J_1(2\sqrt{\pi}|\xi|)$ and the first zero of J_1 is at $3.832\cdots$.)

Fuglede [**Fug74**] was the first to suggest that the disk is not spectral, but the argument was unclear. The situation has since been clarified in the papers of Iosevich, Katz and Pedersen [**IKP99**], who proved that the ball in any dimension is not spectral, and of Iosevich, Katz and Tao [**IKT01**], in which a much more general result is proved: every smooth convex hypersurface cannot have an interior which is a spectral domain. It was also shown by Fuglede [**Fug01**] (for the Euclidean ball in $\mathbb{R}^d$) and by Iosevich and Rudnev [**IR02**] (for any smooth convex body in $\mathbb{R}^d$, for $d\ne 1 \bmod 4$) that there can only be a finite number of orthogonal exponentials in the corresponding L^2 spaces.

The method shown in this section is still interesting because of its simplicity and, perhaps, entertaining as the fact that it works appears to be an accident.

The Fourier Transform of D is radial, as is the function itself, hence the set of zeros of the Fourier Transform is a set of circles centered at the origin. Let r_0 be the radius of the smallest such circle. By a simple numerical calculation we locate $r_0=1.08098\cdots$. Suppose now that the disk is spectral with spectrum Λ. Since $\Lambda-\Lambda\subseteq\{\widehat{\chi_D}=0\}\cup\{0\}$ it follows that $|\lambda-\mu|\ge r_0$ for any $\lambda,\mu\in\Lambda$, $\lambda\ne\mu$, and hence, if we center a copy of a disk of radius $r_0/2$, call it D_1, at each point of Λ, we have a packing of the plane with congruent disks (see Figure 14). The density of such a packing is at most $\pi/\sqrt{12}$, by Fact 1 above.

Since the integral of the *power spectrum* $|\widehat{\chi_D}|^2$ of χ_D is 1 (Parseval), and the power spectrum tiles with Λ it follows that the density of Λ is equal to 1 as well, hence the density of the packing $D_1+\Lambda$ is equal to the area of D_1, which is $\pi r_0^2/4$. So we have the inequality

$$\pi\frac{r_0^2}{4}\le\frac{\pi}{\sqrt{12}},$$

which implies

$$r_0\le\frac{2}{(12)^{1/4}}=1.0745699\cdots,$$

which is in contradiction with Fact 2 above which states that r_0 is approximately 1.08098.

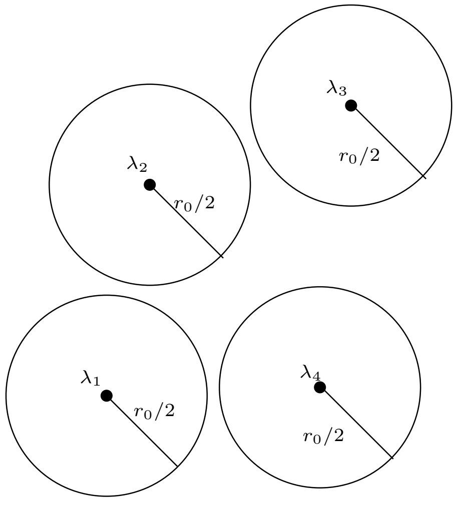


FIGURE 14: The packing by disks of radius $r_0/2$ centered at the points of the spectrum. Thue's result means that the area outside the disks has density $\ge 1 - \pi/\sqrt{12}$

3.5 More results on the Fuglede Conjecture

3.5.1 Convex domains

The convex bodies which tile space have long been known [**V54**, **M80**] to be precisely the polytopes which are symmetric, have symmetric co-dimension one facets and their co-dimension two facets each have a "belt" which consists of four or six facets (the belt of a facet is the collection of all facets of the polytope which are translates of the given facet). It is also known [**M80**] that whenever a convex body Ω tiles space by translation it can also tile by lattice translation. It follows from Theorem 3.2 that convex bodies which tile are also spectral, and possess a lattice spectrum (the dual lattice of their translation lattice).

Our knowledge is much less complete for convex bodies which are spectral. In particular we do not know yet that spectral convex bodies are also tiles, but we are getting there. Most of the results described in this section are in the general direction of showing that well known facts which hold for convex tiles are also true of convex spectral bodies.

In [**IKT01**] it was proved that smooth convex bodies cannot be spectral, a fact which is clearly true of convex bodies which tile, even if one has not heard of the Venkov-McMullen theorem.

Theorem 3.12. (Iosevich, Katz and Tao, 1999)
Suppose that Ω is a symmetric convex body in $\mathbb{R}^d$, $d \ge 2$. If the boundary of Ω is smooth, then it does not admit a spectrum. The same conclusion holds in $\mathbb{R}^2$ if the boundary is piecewise smooth, and has at least one point of non-vanishing Gaussian curvature.

The starting point of the proof is the fact that the zero set

$$\mathcal{Z} = \{\widehat{\chi_\Omega} = 0\}$$

is known, asymptotically, to an ever-higher degree of accuracy. For example, it is a well known fact (see e.g. [**IKT01**]) that if ξ is a zero of $\widehat{\chi_\Omega}$ and $\xi \to \infty$ such that ξ remains inside a cone

$$\mathcal{C} = \left\{\xi : \frac{\langle \xi, u\rangle}{|\xi|} > 1 - \epsilon\right\},$$

where $u \in S^{d-1}$ is the unit outward normal vector at some point $x \in \partial\Omega$ of positive curvature and $\epsilon > 0$ is sufficiently small, then

$$\|\xi\|_{\Omega^o} = \left(\frac{\pi}{2} + \frac{d\pi}{4}\right) + k\pi + o(1), \quad (\xi \to \infty),$$

where Ω^o is the dual body (which is also smooth), d is the dimension and k is an integer. One then uses the fact that if Λ is a spectrum then $\Lambda - \Lambda \subseteq \mathcal{Z}$ in order to reach a contradiction.

It turns out [**IR02**] that for smooth convex bodies with nowhere vanishing Gaussian curvature (such as the Euclidean ball) much more is true than the fact that there is no complete orthogonal set of exponentials for their L^2 space.

Theorem 3.13. (Iosevich and Rudnev, 2002)
Suppose that Ω is a smooth symmetric convex body in $\mathbb{R}^d$, $d \geq 2$, with nowhere vanishing Gaussian curvature. If $d \neq 1 \bmod 4$ then any set of orthogonal exponentials in $L^2(\Omega)$ is finite. If $d = 1 \bmod 4$ such a set may be infinite only if it is a subset of a one-dimensional lattice.

This has also been proved for the ball in any dimension by Fuglede [**Fug01**].

Finally, in dimension $d = 2$ the Fuglede Conjecture may be considered settled for convex bodies [**IKT02**].

Theorem 3.14. (Iosevich, Katz and Tao, 2002)
The only convex domains in $\mathbb{R}^2$ which are spectral are the parallelograms and the symmetric hexagons (these are the only convex tiles as well).

3.5.2 Polytopes with unbalanced facets

Suppose that Ω is a polytope, not necessarily convex, that tiles space by translation. Suppose also that u is one of its face normals and let $F_1^+, \ldots, F_k^+$ be all its facets with outward normal in the direction of u and let $F_1^-, \ldots, F_l^-$ be the facets with outward normal in the direction of $-u$. One can easily see that we must have

$$\left|F_1^+\right| + \cdots + \left|F_k^+\right| = \left|F_1^-\right| + \cdots + \left|F_l^-\right|.$$

The reason is that in any tiling by translates of Ω the facets F_j^+ can only be "countered" by translates of the facets F_j^-. Applying this for a large region in space one deduces that the total area of the plus-facets must equal that of the minus-facets.

The following result [**KP03**] claims that spectral polytopes have the same property.

Theorem 3.15. (Kolountzakis and Papadimitrakis, 2000)
If Ω is a polytope in $\mathbb{R}^d$ which, for some direction u normal to a facet, has more area with outward normal u than it has with outward normal $-u$, then Ω is not spectral. Clearly it can also not be a tile.

We do not present the proof of this result here. However, the following toy-case is rather instructive. Suppose that we have a polytope Ω which has precisely two facets A and B (see the example in Figure 15) with normals parallel to a certain $u \in S^{d-1}$. Assume that facet A has outward normal u and facet B has $-u$, and that the area of A is not equal to that of B.

We claim that in any semi-infinite tube whose axis is the line $\mathbb{R}u$ and *any* bounded domain as base there are only finitely many points of any spectrum. This is impossible as for any spectrum there is a number R such that in any ball of radius R we can find some point of the spectrum. To show the above claim it is enough to show that any such tube is eventually (that is, near infinity) free from zeros of $\widehat{\chi_\Omega}$, or, what amounts to the same thing, free from zeros of

$$\widehat{\nabla_u \chi_\Omega}(\xi) = 2\pi i \langle \xi, u \rangle \widehat{\chi_\Omega}(\xi).$$

Observe now that $\nabla_u \chi_\Omega$ is a measure supported on the facets of the polytope, which is a constant function on every facet, a constant which depends on the angle the facet is forming with u.

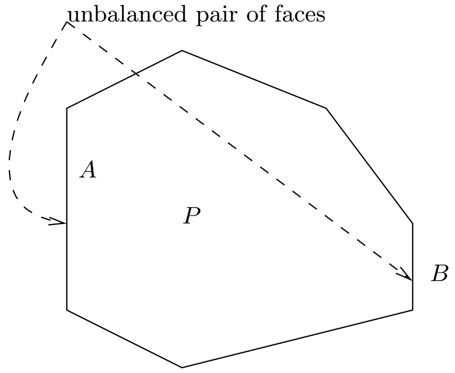


FIGURE 15: A polytope P with many directions of unbalanced faces. The two facets shown are the only ones perpendicular to their normal, yet there is more face measure looking left than looking right. Such a polytope can neither tile by translation nor be spectral.

Look then at what happens to the Fourier Transform $\widehat{\nabla_u \chi_\Omega}$ along the line $\mathbb{R}u$. Along that line the values of the Fourier Transform that we are reading are just the values of the *one-dimensional* Fourier Transform of the *projection* of the measure $\nabla_u \chi_\Omega$ on the line $\mathbb{R}u$. This is the measure μ defined by

$$\mu(E) = \nabla_u \chi_\Omega(E + u^\perp), \quad (E \subseteq \mathbb{R}),$$

and it is clear that μ has a continuous part coming from all the facets which are non-orthogonal to u and also contains the two point masses $|A|\delta_a$ and $-|B|\delta_b$, where $a, b \in \mathbb{R}$ are the points on $\mathbb{R}u$ where the facets A and B project. By the Riemann-Lebesgue lemma the contribution to $\widehat{\mu}$ of the continuous part of μ fades to 0 as we tend to ∞ and it is the Fourier Transform of the atomic part that dominates $\widehat{\mu}$, namely (as $t \to \infty$)

$$\widehat{\mu}(t) \sim |A|e^{2\pi i\langle a,t\rangle} - |B|e^{2\pi i\langle b,t\rangle}$$

whose absolute value is $\geq ||A| - |B||$. So, for large t, there are no zeros on the line, and with a little more care, we can show that the same (albeit farther away) is true in any tube around this line.

3.5.3 Dimension 1

Even in dimension 1 the Fuglede Conjecture appears to be rather hard. The number-theoretic aspect of the problem is seen more clearly here, especially if one looks just at sets of the type

$$\Omega = A + (0, 1), \quad (A \text{ a finite subset of } \mathbb{Z}).$$

The conjecture is still open for this class of sets.

The following are interesting partial results.

1. Łaba [**Łab01**] showed that whenever $|A| = 2$ the conjecture is true.

2. This is improved to $|A| = 3$ by Łaba in [**Łab02**]. In the same paper it is also shown that if $|A|$ has at most two prime factors then if Ω is a tile it is also spectral.

3. Łaba also shows in [**Łab02**] that if $|A| > 3/2(\max A - \min A)$ then the Ω is a tile if and only if it is spectral. This is generalized by Kolountzakis and Łaba [**KŁ01**] to any set Ω of measure 1 which is a subset of $(0, 3/2 - \epsilon)$, for some $\epsilon > 0$. In fact what is really shown in [**KŁ01**] is that such "tight" domains can only be spectral or tiles of they tile by the lattice $\mathbb{Z}$.

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