IMRN, to appear

Multi-lattice tiles

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March 1997; revised July 1997

Abstract

Let $\Lambda_0, \ldots, \Lambda_n \subset \mathbf{R}^d$ be a finite collection of lattices of the same volume. When does there exist a set $\Omega \subset \mathbf{R}^d$ that is a fundamental domain for all Λ_i, $i = 0, \ldots, n$? The main result of this paper is that when the (group-theoretic) sum $\Lambda_0^* + \cdots + \Lambda_n^*$ of the dual lattices is direct (this means that the equation $x_0 + \cdots + x_n = 0$, with $x_i \in \Lambda_i^*$, has no non-trivial solution), then a Borel *measurable* $\Omega \subset \mathbf{R}^d$ exists which is almost a fundamental domain for all the Λ_i, i.e., it covers almost all cosets mod each Λ_i exactly once.

We prove and use a theorem of the Kronecker-Weyl type that describes when a lattice is sumultaneously dense modulo a collection of other lattices in $\mathbf{R}^d$.

§0. Introduction

Let $\Lambda_0, \ldots, \Lambda_n \subset \mathbf{R}^d$ be a finite collection of lattices of the same volume. When does there exist a set $\Omega \subset \mathbf{R}^d$ that is a fundamental domain for all Λ_i, $i = 0, \ldots, n$? The main result of this paper is that when the (group-theoretic) sum $\Lambda_0^* + \cdots + \Lambda_n^*$ of the dual lattices is direct (this means that the equation $x_0 + \cdots + x_n = 0$, with $x_i \in \Lambda_i^*$, has no non-trivial solution), then a Borel *measurable* $\Omega \subset \mathbf{R}^d$ exists which is almost a fundamental domain for all the Λ_i, i.e., it covers almost all cosets mod each Λ_i exactly once.

The set Ω, which is generally unbounded, can also be viewed as a "tile" that tiles $\mathbf{R}^d$ by translation with any one of the lattices $\Lambda_0, \ldots, \Lambda_n$.

0.1 Fundamental domains

All groups that appear in this paper are abelian. Let G be a subgroup of H. A set $F \subseteq H$ is called a *fundamental domain* (FD) of G in H (or a *transversal* of G in H) if it contains exactly one element from each coset of H/G. We shall be interested in the case when G is a lattice of full rank in $H = \mathbf{R}^d$.

Any such lattice Λ can be written (in many ways) as $\Lambda = A\mathbf{Z}^d$, where $A \in M_d(\mathbf{R})$ is a non-singular matrix. From such a representation one may construct a fundamental domain F by taking the parallelepiped $A[0,1)^d$. This is of course a Borel measurable set. Furthermore, by elementary volume considerations, one can show that any measurable FD of Λ has to have volume $\det A$.

A set Ω that contains exactly one point out of almost all (Lebesgue) cosets of $\mathbf{R}^d/\Lambda$ will be called an *almost*-FD for Λ in $\mathbf{R}^d$. More specifically, there exists a set $E \subset \mathbf{R}^d$ of measure 0 such that for any $x \notin E$ the coset $x + \Lambda$ contains precisely one point of Ω.

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Partially supported by NSF grant DMS 9706753

0.2 A problem of Steinhaus

The motivation for the results of this paper comes from the following problem that was posed by Steinhaus [**Mo**, problem 59]:

> Does there exist a set $\Omega \subseteq \mathbf{R}^2$ which, no matter how translated and rotated, contains exactly one point in $\mathbf{Z}^2$?

Otherwise stated, it is asked whether there exists a set Ω that is a FD for each lattice in the collection

$$\mathcal{G} = \{R_\theta \mathbf{Z}^2 : \ \theta \in [0, 2\pi)\}, \tag{1}$$

where the matrix R_θ represents rotation by θ about the origin.

At least in the case when one demands the set Ω to be a measurable almost-FD for all the lattices in $\mathcal{G}$ (or even a countable "dense" subset of $\mathcal{G}$) there have been many results (Sierpiński [**Si**], Croft [**Cr**], Beck [**Be**], and Kolountzakis [**K1**, **K2**]) in the direction that certain classes of sets cannot have this *Steinhaus property* (typically, that a set whose area decreases too fast near infinity cannot have the Steinhaus property). Ciucu [**Ci**] proves that no closed subset of the plane (bounded or not) can have the Steinhaus property. Komjáth [**Kom**] proves the existence of such sets if one replaces the group $\mathbf{Z}^2$ of (1) with the group $\mathbf{Z}$ (embedded in $\mathbf{R}^2$ as a subset of, say, the x-axis) or the group $\mathbf{Q}^2$. (These sets are certainly non-measurable in the case of rotations of $\mathbf{Q}^2$, as any FD of a non-discrete countable subgroup of the plane cannot be measurable. The FD of $\mathbf{R}/\mathbf{Q}$ is, in fact, the first example of a non-measurable set that one encounters.)

0.3 Common FDs for a finite set of lattices

It is then reasonable to relax Steinhaus's demands and ask whether there exists a common FD Ω for any finite subset of $\mathcal{G}$. In this paper we investigate this with the requirement that Ω be measurable. In the language of the Steinhaus problem, we want to find a set $\Omega \subset \mathbf{R}^2$ for a given set of directions $\theta_0, \ldots, \theta_n \in [0, 2\pi)$ which, no matter how translated in any of these orientations, contains exactly one lattice point.

There is no reason that we should restrict ourselves to the case of a finite collection of lattices in $\mathbf{R}^2$ all of which are rotations of a fixed lattice ($\mathbf{Z}^2$ in the case of Steinhaus's problem). So, from now on, let $\mathcal{G} = \{\Lambda_0, \ldots, \Lambda_n\}$ be an arbitrary collection of lattices in $\mathbf{R}^d$, $d \geq 2$, of volume 1 (one cannot expect to find a measurable common FD for them unless they have the same volume).

We shall prove that for a "generic" such collection $\mathcal{G}$ there is indeed a measurable common FD.

0.4 The problem as a problem of lattice tiling

Let $f \in L^1(\mathbf{R}^d)$ and $\Lambda \subset \mathbf{R}^d$ be a lattice. We say that f tiles $\mathbf{R}^d$ with Λ if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = C, \quad \text{for almost every } x \in \mathbf{R}^d, \tag{2}$$

where C (the *weight* of the tiling) is a constant and the sum converges absolutely. The most usual situation is when f is the indicator function of a "nice" set $\Omega \subset \mathbf{R}^2$ and $C = 1$. It is an easy application of the Poisson Summation Formula (see [**K1**, **K2**]) that (2) holds for some C if and only if the Fourier Transform $\widehat{f}$ of f (see (3)) vanishes on $\Lambda^* \setminus \{0\}$, where Λ^* is the lattice dual to Λ (see §2.2 for the definition).

In this language of tilings the set Ω is a common almost-FD for the collection $\mathcal{G}$ of lattices if and only if its indicator function $\mathbf{1}_\Omega$ tiles with each $\Lambda \in \mathcal{G}$ with weight 1.

Let us remark that if one wants to find a tile f for all $\Lambda \in \mathcal{G}$ that is not necessarily an indicator function then it is very easy to do so by just taking a function f whose Fourier Transform

$$\widehat{f} : \mathbf{R}^d \to \mathbf{C}, \quad \widehat{f}(\xi) = \int_{\mathbf{R}^d} f(x) e^{-2\pi i \langle x, \xi \rangle} \, dx, \tag{3}$$

vanishes outside a sufficiently small neighborhood of 0, takes the value 1 at 0, and is smooth (see [**K1**, **K2**]). This works even if $\mathcal{G}$ is the infinite family in the Steinhaus problem (1).

On the other hand, if one just wants to construct a function $f \in L^1(\mathbf{R}^d)$ that tiles with *any* set of lattices $\Lambda_0, \ldots, \Lambda_n \subset \mathbf{R}^d$ (perhaps with a different weight for each as these lattices are not required to have the same volume) one merely has to write

$$f = \mathbf{1}_{D_0} \star \cdots \star \mathbf{1}_{D_n}, \quad \text{(convolution)} \tag{4}$$

where $\mathbf{1}_{D_i}$ denotes the indicator function of any measurable fundamental domain D_i (say some fundamental parallelepiped) of the lattice Λ_i. That is so because if some function f tiles with a lattice (or any other discrete set) then so does $f \star g$, where $g \in L^1(\mathbf{R}^d)$ is arbitrary.

0.5 New Results

From now on we shall say that the subgroups $G_0, \ldots, G_n$ of a group X have a *direct sum* if the equation

$$x_0 + \cdots + x_n = 0, \quad (x_i \in G_i) \tag{5}$$

has only the trivial solution.

If one does not ask that the FD Ω is measurable, there is no need for the lattices Λ_j in $\mathbf{R}^d$ to have the same volume. In fact, we have the following more general result [**Lou**].

Theorem 1 *Let the subgroups $G_0, \ldots, G_n$ of a group X all have the same cardinality, and let the sum $G_0 + \cdots + G_n \subseteq X$ be direct. Then there is a common FD for the groups $G_0, \ldots, G_n$ in X.*

Proof. It is sufficient to produce a common FD F for the G_i in the group $\sum_{i=0}^n G_i$. Because then, if F' is a FD of $\sum_{i=0}^n G_i$ in X the set $F + F'$ is a common FD for the G_i in X. In fact, it is also true that if one has a common FD of the G_i in X then one can get, by intersecting it with $\sum_{i=0}^n G_i$, a common FD of the G_i in $\sum_{i=0}^n G_i$. This means that it is always enough to look for a common FD of the G_i in their sum and not in any larger group.

To construct the common FD of the G_i in $\sum_{i=0}^n G_i$, enumerate $G_i = \left\{ g_j^i : \ j \in J \right\}$, where J is some index set, and define the set F to consist of all expressions

$$S = \sum_{i=1}^n (g_{j_i}^0 + g_{j_i}^i), \quad (j_1, \ldots, j_n \in J). \tag{6}$$

To show that F is a common FD for the G_i we must show that, for given $a_0, \ldots, a_n \in J$ and $i = 0, \ldots, n$, the condition

$$S - (g_{a_0}^0 + \cdots + g_{a_n}^n) \in G_i \tag{7}$$

has a unique solution in $j_1, \ldots, j_n$. For $i = 0$ (7) has the solution $j_1 = a_1, \ldots, j_n = a_n$, while for, say, $i = 1$ (7) has the solution $j_2 = a_2, \ldots, j_n = a_n$ and j_1 is determined by

$$g_{j_1}^0 + g_{a_2}^0 + \cdots g_{a_n}^0 - g_{a_0}^0 = 0.$$

□

(Let us give a simple example [**Lou**] where a common FD does not exist. Take $X = \mathbf{Z}_2 \times \mathbf{Z}_2$ and let G_0, G_1, G_2 be the three copies of $\mathbf{Z}_2$ in X. These groups do have the same index in X, which is clearly a necessary condition, but they do not have a common FD in X.)

But, although it is a simple matter to construct a common FD, call it F, of the lattices $\Lambda_i \subset \mathbf{R}^2$ in the sum $\sum_{i=0}^n \Lambda_i$ when this sum is direct, notice that if this happens then, necessarily, the group $\sum_{i=0}^n \Lambda_i$ is a countable *non-discrete* subgroup of $\mathbf{R}^d$ (in fact, it is everywhere dense). So, when one tries to use F in order to find a common FD of the Λ_i in the larger group $\mathbf{R}^d$ (in the way that was indicated in the proof of Theorem 1), one runs into the non-measurability of any FD F' of $\sum_{i=0}^n \Lambda_i$ in $\mathbf{R}^d$. So this method cannot guarantee the measurability of the resulting common FD, which, in fact, will certainly be non-measurable when not all the volumes of the Λ_i are equal.

The main construction regarding measurable common almost-FDs takes place in §1. There we construct a common almost-FD for a finite collection of lattices of $\mathbf{R}^d$ that has the following density property.

Property A. *We shall say that a collection of lattices $\Lambda_0, \ldots, \Lambda_n \subset \mathbf{R}^d$ has Property A if for each $\epsilon > 0$ and for each $x_0, \ldots, x_n \in \mathbf{R}^d$ there exist $\lambda_0 \in \Lambda_0, \ldots, \lambda_n \in \Lambda_n$, with $|\lambda_j|$ arbitrarily large, such that*

$$|x_i - \lambda_i - (x_j - \lambda_j)| \le \epsilon, \quad \textit{for all } i, j = 0, \ldots, n. \tag{8}$$

That is, we can get any collection of points $x_0, \ldots, x_n \in \mathbf{R}^d$ arbitrarily close to each other by translating x_i by some $\lambda_i \in \Lambda_i$, $i = 0, \ldots, n$.

In §1 we prove:

Theorem 2 *If the lattices $\Lambda_0, \ldots, \Lambda_n \subset \mathbf{R}^d$ all have the same volume and have Property A then they possess a Borel measurable common almost-FD (which is generally unbounded).*

In §2 we characterize the collections $\Lambda_0, \ldots, \Lambda_n$ that have Property A. These are precisely the collections for which the sum of the dual lattices

$$\Lambda_0^* + \cdots + \Lambda_n^* \tag{9}$$

is direct.

The following is an immediate implication of the completeness of Lebesgue measure.

Theorem 3 *If the collection of lattices $\Lambda_0, \ldots, \Lambda_n \subseteq \mathbf{R}^d$ have a not necessarily measurable common FD Ω' and a measurable common almost-FD Ω'' then they have a measurable common FD Ω (no exceptional set).*

We cannot conclude however that Ω is Borel measurable.

Proof. Let Ω'' contain exactly one element out of every coset $x + \Lambda_j$, $j = 0, \ldots, n$, for all $x \notin E$, where $E \subset \mathbf{R}^d$ is an exceptional set of measure 0.

Define

$$T = E + \sum_{i=0}^{n} \Lambda_i$$

and observe that $\mu(T) = 0$, since $\sum_{i=0}^{n} \Lambda_i$ is countable, and that T is a union of cosets of Λ_i, for all i.

Finally let

$$\Omega = (\Omega'' \setminus T) \cup (\Omega' \cap T).$$

The measurability of Ω follows from the fact that $\mu(T) = 0$ and the completeness of Lebesgue measure (i.e., every subset of a set of measure 0 is measurable).

Now fix i. From each coset of Λ_i in T the FD Ω' picks out exactly one representative. From each coset of Λ_i disjoint from T the set Ω'' picks out exactly one element, since it is an almost-FD for Λ_i and all exceptional cosets are contained in T. It follows that Ω is a common FD for the collection $\Lambda_0, \ldots, \Lambda_n$. $\Box$

Combining Theorems 2 and 3 we immediately get

Theorem 4 *If the lattices $\Lambda_0, \ldots, \Lambda_n \subset \mathbf{R}^d$ have the same volume and both $\Lambda_0 + \cdots + \Lambda_n$ and $\Lambda_0^* + \cdots + \Lambda_n^*$ are direct sums, then they have a Lebesgue measurable common FD $\Omega \subset \mathbf{R}^d$.*

Let us point out here that the conditions "$\Lambda_0 + \cdots + \Lambda_n$ is direct" and "$\Lambda_0^* + \cdots + \Lambda_n^*$ is direct" are not the same, even in dimension 1 [**Lou**]. In dimension 1 the dual lattice of $\alpha\mathbf{Z}$ is $\alpha^{-1}\mathbf{Z}$ and the collection of lattices $\alpha_0\mathbf{Z}, \ldots, \alpha_n\mathbf{Z}$ has a direct sum if and only if the numbers $\alpha_0, \ldots, \alpha_n$ are $\mathbf{Q}$-linearly independent. The three numbers

$$2^{1/3},\ 3^{1/3},\ \frac{1}{2^{-1/3} + 3^{-1/3}}$$

are $\mathbf{Q}$-linearly independent (a non-trivial vanishing $\mathbf{Q}$-linear combination of them gives rise to a quadratic polynomial with rational coefficients and with $(2/3)^{1/3}$ as a root, which is impossible) while their reciprocals

$$2^{-1/3},\ 3^{-1/3},\ 2^{-1/3} + 3^{-1/3}$$

are clearly dependent over $\mathbf{Q}$.

§1. The construction of a measurable common almost-FD

Here we prove Theorem 2. The common almost-FD $\Omega \subseteq \mathbf{R}^d$ that we construct is a countable union of disjoint closed polyhedra (in fact, rectangles).

The letter C will stand in this section for a positive constant that may not depend on the parameter $K \to \infty$ and this constant is not necessarily the same in all its occurences.

The lattices Λ_j, $j = 0, \ldots, n$, are given by

$$\Lambda_j = A_j \mathbf{Z}^d, \quad \det A_j = 1. \tag{10}$$

Let D_j be the standard FD for the lattice Λ_j, i.e.,

$$D_j = A_j [0,1)^d, \tag{11}$$

which is a parallelepiped of volume 1.

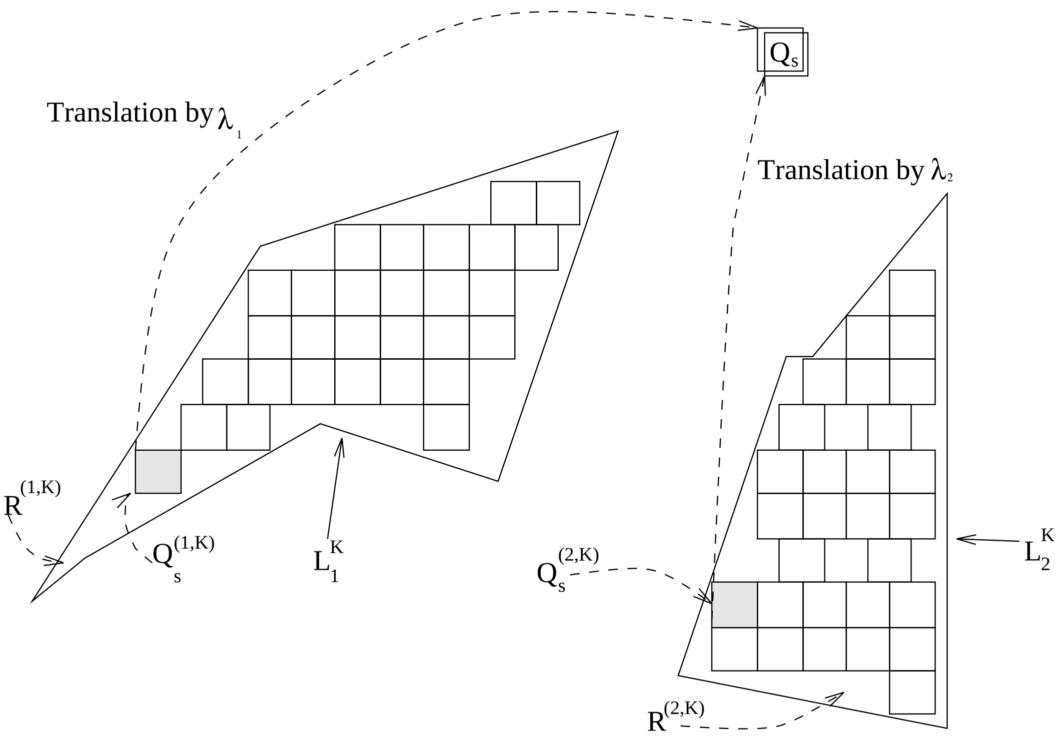

Figure 1: Construction of the common almost-FD for two lattices, $d = 2$

Let $\Omega_0 = \emptyset$. In the end we shall have

$$\Omega = \bigcup_{k=1}^{\infty} \Omega_k,$$

where the K-th approximation

$$A_K = \bigcup_{k=1}^{K} \Omega_k$$

has measure $\mu(A_K) \to 1$, as $K \to \infty$, and for each $j = 0, \ldots, n$ almost all cosets $x + \Lambda_j$ have no more than one point in A_K. It follows that Ω contains exactly one element from almost all the cosets of Λ_j, for each $j = 0, \ldots, n$, and is therefore a common almost-FD for the collection $\Lambda_0, \ldots, \Lambda_n$.

Assume that we have already defined $\Omega_0, \ldots, \Omega_K$. The set Ω_{K+1} will be defined as follows. The "projection" $\pi_j : \mathbf{R}^d \to D_j$ is defined by the relation

$$x - \pi_j(x) \in \Lambda_j.$$

The "leftover" after stage K is then defined by

$$L_j^{(K)} = D_j \setminus \pi_j(A_K), \quad \text{for } j = 0, \ldots, n. \qquad (12)$$

We have to ensure that $\mu\left(L_j^{(K)}\right) \to 0$, as $K \to \infty$.

Our construction will guarantee that each of the leftovers $L_j^{(K)}$ consists of a finite collection of polyhedra. Choose $\epsilon > 0$ to be so small so as to be able to write

$$L_j^{(K)} = \left(\bigcup_{s=1}^{S} Q_s^{(j,K)}\right) \cup R^{(j,K)}, \quad (j = 0, \ldots, n) \qquad (13)$$

where the $Q_s^{(j,K)}$, $s = 1, \ldots, S = S(K)$, are axis-aligned, closed cubes with disjoint interiors of side ϵ, and

$$\mu\left(R^{(j,K)}\right) \le \frac{1}{K}. \tag{14}$$

Notice that the same number $S = S(K)$ of cubes is used independently of j. (The conctruction is shown for two lattices in Figure 1 in dimension $d = 2$.)

For each $s = 1, \ldots, S$, let $c_s^{(j,K)}$ be the center of the cube $Q_s^{(j,K)}$ and, using Property A, define $\lambda_s^{(j,K)} \in \Lambda_j$ to be such that all

$$c_s^{(j,K)} - \lambda_s^{(j,K)}, \quad j = 0, \ldots, n,$$

are at most $\frac{\epsilon}{K}$ apart. The $\lambda_s^{(j,K)}$ are also taken large enough so that, for fixed j, no two translated cubes $Q_s^{(j,K)} - \lambda_s^{(j,K)}$ overlap.

Consider then the intersection of the $n+1$ translated cubes

$$\widetilde{Q}_s^{(K)} = \bigcap_{j=0}^{n} \left(Q_s^{(j,K)} - \lambda_s^{(j,K)}\right) \tag{15}$$

and notice that

$$\mu(\widetilde{Q}_s^{(K)}) \ge \epsilon^d - C\frac{\epsilon^d}{K}. \tag{16}$$

Define

$$\Omega_{K+1} = \bigcup_{s=1}^{S} \widetilde{Q}_s^{(K)}.$$

We have $L_j^{(K+1)} = L_j^{(K)} \setminus \pi_j(\Omega_{K+1})$ and

$$\mu\left(L_j^{(K)}\right) \to 0,$$

as $K \to \infty$. This is so because $L_j^{(K)} \setminus \pi_j(\Omega_{K+1})$ consists of the sets $R^{(j,K)}$, $j = 0, \ldots, n$, which have total measure $\le \frac{n+1}{K}$ plus a set of measure $C\frac{\epsilon^d}{K}$ for each $s = 1, \ldots, S$, which amounts to no more than $\frac{C}{K}$ of measure, as clearly $\epsilon^d S \le 1$. □

§2. A Kronecker-Weyl theorem

2.1 When Property A holds

In this section we give a necessary and sufficient condition on the collection of lattices in $\mathbf{R}^d$

$$\mathcal{G} = \{\Lambda_0, \ldots, \Lambda_n\}$$

to have Property A of §0.5. Assume that the lattices Λ_j have a representation

$$\Lambda_j = A_j \mathbf{Z}^d,$$

where $A_j \in M_d(\mathbf{R})$ is non-singular. (We do not assume in this section that the lattices have the same volume.)

First we restate Property A in a slightly different way, which distinguishes the first lattice Λ_0. *For every $\epsilon > 0$ and for each collection $x_1, \ldots, x_n$ there are $\lambda \in \Lambda_0$ and $\lambda_1 \in \Lambda_1, \ldots, \lambda_n \in \Lambda_n$, all of them of arbitrarily large modulus, such that*

$$|x_1 - (\lambda_1 - \lambda)| \le \epsilon, \ldots, |x_n - (\lambda_n - \lambda)| \le \epsilon.$$

In other words, given any "pattern" of points $x_0 = 0, x_1, \ldots, x_n \in \mathbf{R}^d$, we can translate it somewhere so that there is a point of Λ_i ϵ-close to the translate of x_i, $i = 0, \ldots, n$.

Consider now the lattice

$$\Gamma = \Lambda_1 \times \cdots \times \Lambda_n \subset \mathbf{R}^{nd},$$

and the discrete subgroup of $\mathbf{R}^{nd}$

$$G = \mathrm{diag}(n, \Lambda_0) = \{(\lambda, \ldots, \lambda) : \ \lambda \in \Lambda_0\}$$

(which is not of full rank in $\mathbf{R}^{nd}$ unless $n = 1$). Let $D = D_1 \times \cdots \times D_n$ be a fundamental parallelepiped of Γ, where D_j is the fundamental parallelepiped of Λ_j given by $D_j = A_j[0,1)^d$. Having fixed this FD of Γ we can now unambiguously speak of the point $x \bmod \Gamma \in D$ for each $x \in \mathbf{R}^{nd}$.

It is now clear that Property A is equivalent to the set ($G \bmod \Gamma$) being dense in D. When this happens is characterized by the following theorem.

Theorem 5 **(Kronecker-Weyl)** *The following three statements are equivalent.*

(i) The equation $\lambda_0^* + \ldots + \lambda_n^* = 0$, with $\lambda_j^* \in \Lambda_j^*$, has only the trivial solution. In other words, the sum $\Lambda_0^* + \cdots + \Lambda_n^*$ is direct.

(ii) $G \bmod \Gamma$ is dense in D. That is, the lattices $\Lambda_0, \ldots, \Lambda_n$ have Property A.

(iii) $G \bmod \Gamma$ is uniformly distributed (u.d.) in D.

2.2 Remarks

1. **Dual lattices.** If $\Lambda = A\mathbf{Z}^d \subset \mathbf{R}^d$ is a lattice then its dual lattice Λ^* is defined as follows:

$$\Lambda^* = \left\{ x \in \mathbf{R}^d : \ \ \forall \lambda \in \Lambda : \ \langle x, \lambda \rangle \in \mathbf{Z} \right\}.$$

One can show that $\Lambda^* = A^{-\top}\mathbf{Z}^d$.

2. **Harmonic analysis on $\mathbf{R}^d/\Lambda$.** Let D be a fundamental paralellepiped of the lattice $\Lambda \subset \mathbf{R}^d$. We view $D = \mathbf{R}^d/\Lambda$ as a compact abelian group. The continuous characters of that group are the Λ-periodic functions

$$x \to \exp 2\pi i \langle \lambda^*, x \rangle, \tag{17}$$

where $\lambda^* \in \Lambda^*$.

3. **Uniform distribution.** For a lattice $\Gamma \subset \mathbf{R}^N$ with fundamental parallelepiped D (of volume 1) and a sequence of points $P_{k_1,\ldots,k_s} \in \mathbf{R}^N$, parametrized by $k_j \in \mathbf{Z}$, we shall say that $P_{k_1,\ldots,k_s}$ is u.d. mod Γ if for every rectangle $I \subset D$

$$\lim_{R\to\infty} \frac{1}{(2R+1)^s} \#\{(k_1, \ldots, k_s) \in \{-R, \ldots, R\}^s : \ \ P_{k_1,\ldots,k_s} \bmod \Gamma \in I\} \ \ = \ \ |I|.$$

Alternatively, the set of points $P_{k_1,\ldots,k_s}$ is u.d. mod Γ if and only if for every continuous Γ-periodic function $f : \mathbf{R}^N \to \mathbf{C}$ we have

$$\lim_{R\to\infty} \frac{1}{(2R+1)^s} \sum_{-R\le k_j\le R} f(P_{k_1,\ldots,k_s}) \ = \ \int_D f(x)\, dx. \tag{18}$$

This definition does not depend on the choice of the fundamental parallelepiped D.

2.3 Proof of Theorem 5

Notice first that (iii) $\Longrightarrow$ (ii) trivially.

To see why (ii) $\Longrightarrow$ (i), assume that there are some $\lambda_0^* \in \Lambda_0^*, \lambda_1^* \in \Lambda_1^*, \ldots, \lambda_n^* \in \Lambda_n^*$, not all zero, such that

$$\lambda_0^* + \lambda_1^* + \cdots + \lambda_n^* = 0. \tag{19}$$

Define the non-constant Γ-periodic function $f : \mathbf{R}^{nd} \to \mathbf{C}$ by

$$f(x_1,\ldots,x_n) = \exp 2\pi i\,(\langle\lambda_1^*, x_1\rangle + \cdots + \langle\lambda_n^*, x_n\rangle), \quad x_j \in \mathbf{R}^d.$$

Notice now that, for $\lambda \in \Lambda_0$,

$$\begin{aligned} f(\lambda,\ldots,\lambda) &= \exp 2\pi i\langle\lambda_1^* + \cdots + \lambda_n^*, \lambda\rangle \\ &= \exp 2\pi i\langle -\lambda_0^*, \lambda\rangle \quad \text{(from (19))} \\ &= 1, \end{aligned}$$

which is incompatible with $G = \mathrm{diag}(n, \Lambda_0)$ being dense mod Γ.

It remains to show that (i) $\Longrightarrow$ (iii). We use the characterization of uniform distribution using integrals (see Remark 3 above), and for simplicity we assume that Γ has volume 1. The continuous characters of the group $\mathbf{R}^{nd}/\Gamma$ are the functions

$$\phi_{\lambda_1^*,\ldots,\lambda_n^*}(x_1,\ldots,x_n) = \exp 2\pi i\,(\langle\lambda_1^*, x_1\rangle + \cdots + \langle\lambda_n^*, x_n\rangle), \tag{20}$$

indexed by $\lambda_1^* \in \Lambda_1^*, \ldots, \lambda_n^* \in \Lambda_n^*$, and it is well known that the linear combinations of those characters are dense in $C(\mathbf{R}^{nd}/\Gamma)$. Therefore it suffices to verify (18) with f being any continuous character.

So let $f(x) = \phi_{\lambda_1^*,\ldots,\lambda_n^*}(x_1,\ldots,x_n)$ for some fixed $\lambda_1^* \in \Lambda_1^*, \ldots, \lambda_n^* \in \Lambda_n^*$.

We may assume that $\lambda_1^*, \ldots, \lambda_n^*$ are not all 0 and write $\Lambda_0 = A\mathbf{Z}^d$. We have

$$\begin{aligned} \sum_{m\in\{-R,\ldots,R\}^d} \phi_{\lambda_1^*,\ldots,\lambda_n^*}(Am,\ldots,Am) &= \sum_{m\in\{-R,\ldots,R\}^d} \exp 2\pi i\langle\sum_{i=1}^n \lambda_i^*, Am\rangle \\ &= \sum_{m\in\{-R,\ldots,R\}^d} \exp 2\pi i\langle A^\top \sum_{i=1}^n \lambda_i^*, m\rangle \\ &= \prod_{k=1}^d \sum_{m=-R}^R \exp 2\pi i m\langle A^\top \sum_{i=1}^n \lambda_i^*, e_k\rangle, \end{aligned}$$

where $e_k \in \mathbf{R}^d$ is the k-th element of the standard basis. Clearly we cannot have $A^\top \sum_{i=1}^n \lambda_i^* \in \mathbf{Q}^d$ as this would imply $M\sum_{i=1}^n \lambda_i^* = A^{-\top}v$, for some $v \in \mathbf{Z}^d$, $M \in \mathbf{Z}$, and the right hand side is an element of Λ_0^* (this would thus contradict the direct sum hypothesis). So one of the factors in the product above, say the factor that corresponds to $k=1$, has $\left\langle A^\top \sum_{i=1}^n \lambda_i^*, e_1\right\rangle$ irrational. For this factor we have

$$\sum_{m=-R}^{R} \exp 2\pi i m \langle A^\top \sum_{i=1}^{n} \lambda_i^*, e_1\rangle = \exp(-2\pi i R \langle A^\top \sum_{i=1}^{n} \lambda_i^*, e_1\rangle) \times \times \frac{\exp 2\pi i (2R+1)\langle A^\top \sum_{i=1}^n \lambda_i^*, e_1\rangle - 1}{\exp 2\pi i \langle A^\top \sum_{i=1}^n \lambda_i^*, e_1\rangle - 1},$$

which is bounded in absolute value by a constant independent of R. And all the other $d-1$ factors are bounded above in absolute value by $2R+1$. We conclude that

$$\lim_{R\to\infty} (2R+1)^{-d} \sum_{m\in\{-R,\ldots,R\}^d} \phi_{\lambda_1^*,\ldots,\lambda_n^*}(Am,\ldots,Am) = 0 = \int_D \phi_{\lambda_1^*,\ldots,\lambda_n^*}.$$

□

§3. Some questions

The following open problems arise naturally from this work.

3.1 The commensurable case

What can we say about the lattices Λ_i, $i=0,\ldots,n$, admitting a common FD when there are non-trivial solutions to the equation

$$\lambda_0 + \cdots + \lambda_n = 0, \quad (\lambda_i \in \Lambda_i)?$$

Can the Λ_i possess a measurable common almost-FD when the sum $\sum_{i=0}^n \Lambda_i^*$ is not direct?

The following example shows that there are cases of commensurable lattices which possess no common almost-FD in a strong sense. The study of this example was suggested by the anonymous referee because of its resemblance to the example of the three subgroups of order 2 in $\mathbf{Z}_2\times\mathbf{Z}_2$ which do not have a common FD. Let

$$\Lambda_0 = (2\mathbf{Z})\times\mathbf{Z},\ \Lambda_1 = \mathbf{Z}\times(2\mathbf{Z}),\ \text{and}\ \Lambda_2 = \left\{(k,l)\in\mathbf{Z}^2:\ k = l \bmod 2\right\}.$$

It is easy to see that

$$\mathbf{Z}^2 = \sum_{i=0}^{2} \Lambda_i = \bigcup_{i=0}^{2} \Lambda_i.$$

Suppose now that $\Omega\subset\mathbf{R}^2$ is such that for all $x\in\mathbf{R}^2$, outside a set E of measure 0, we have that $x+\Lambda_i$ contains exactly one point of Ω, for all $i=0,1,2$. (We do not assume that Ω is measurable.)

It follows that for almost all $x\in\mathbf{R}^2$ (with an exceptional set perhaps different from E) we have

$$\left|(x+\mathbf{Z}^2)\cap\Omega\right| = 2 \text{ and } |(x+\Lambda_i)\cap\Omega| = 1,\ i=0,1,2.$$

(Indeed, $\mathbf{Z}^2$ is the disjoint union of Λ_0 and $\Lambda_0+(1,0)$ and so are all its translates. We define the set

$$E' = E\cup(E-(1,0)),$$

which is clearly still a null set. Then, for $x \notin E'$ the set $x + \mathbf{Z}^2$ contains exactly two points of Ω, since the two disjoint copies of Λ_0 therein both contain exactly one Ω-point.)

By translating Ω we may assume that this holds for $x = 0$. Let then

$$\{z, w\} = (x + \mathbf{Z}^2) \cap \Omega.$$

This implies that $w - z$ is in some Λ_i, $i = 0, 1, 2$, since $\mathbf{Z}^2$ is the *union* of the Λ_i. This is of course a contradiction since z or w already belong to the same Λ_i. Hence the Λ_i have no common FD in $\mathbf{R}^2$ in a strong sense.

3.2 Bounded domain?

Can we construct a bounded, measurable common almost-FD for the Λ_i under the assumptions of Theorem 2?

It is easy to contruct a bounded set-theoretic common FD under the assumption that $\sum_{i=0}^{n} \Lambda_i$ is direct and that the Λ_i have the same volume (I do not know whether this last assumption is necessary–it is certainly used in the proof below). According to Theorem 1, if λ_j^i, $j \in \mathbf{N}$, is an enumeration of the points in Λ_i, then the collection of the points of the type

$$P = \sum_{i=1}^{n} (\lambda_{j_i}^0 + \lambda_{j_i}^i), \quad (j_1, \ldots, j_n \in \mathbf{N}) \tag{21}$$

forms a common FD for the Λ_i, $i = 0, \ldots, n$, in the group $\sum_{i=0}^{n} \Lambda_i$. We have to find a suitable enumeration of the lattices.

First choose an arbitrary enumeration of the lattice Λ_0. The enumerations of $\Lambda_1, \ldots, \Lambda_n$ are chosen so that

$$\left|\lambda_j^0 + \lambda_j^i\right| \le 2\rho, \quad (j \in \mathbf{N}, i = 1, \ldots, n). \tag{22}$$

The value of the constant ρ is such that there is fundamental parallelepipeds D_i of Λ_i, $i = 0, \ldots, n$, which have 0-symmetric closure and diameter at most ρ.

Suppose for the moment that we have achieved (22). To construct a bounded common FD for the Λ_i, let F be the collection of the points P of (21). From (22) we get that the diameter of F is at most $2\rho n$. Let also F' be a FD of $\sum_{i=0}^{n} \Lambda_i$ in $\mathbf{R}^2$ which is bounded. This exists since the group $\sum_{i=0}^{n} \Lambda_i$ is everywhere dense in $\mathbf{R}^2$ and hence one can choose a representative from each coset that belongs, say, to the disk $\{x : |x| < \epsilon\}$, for some given $\epsilon > 0$. The set $F + F'$ is then a common FD of the Λ_i in $\mathbf{R}^d$ and has diemeter at most $2\rho n + \epsilon$.

I owe much of the following simple proof of (22) to P. Papasoglu (see also [**La**] where a more general problem is treated). It is enough to construct a bijection, say, $f : \Lambda_0 \to \Lambda_1$ such that $|\lambda - f(\lambda)| \le 2\rho$. Assume that the fundamental domains D_0 and D_1 have 0-symmetric closures, $\overline{D_0}$ and $\overline{D_1}$, and define $D = \overline{D_0} + \overline{D_1}$. We have that the diameter of D is at most 2ρ.

Let us first construct such an $f : \Lambda_0 \to \Lambda_1$ which is one-to-one, but not necessarily onto. For this we show that there exists a system of distinct reprensentatives (SDR) for the family of finite sets

$$\Lambda_1 \cap (\lambda + D), \quad (\lambda \in \Lambda_0). \tag{23}$$

By compactness it is enough to show that every finite subfamily of (23) admits a SDR. And by P. Hall's "marriage" theorem [**Ca**, Chapter 6] it is enough to show that for any finite $A \subset \Lambda_0$ we have

$$|\Lambda_1 \cap (A + D)| \ge |A|. \tag{24}$$

Clearly the volume of $V = A + \overline{D_0}$ is exactly $|A|$ (assuming the lattices unimodular). The sets $\lambda_1 + \overline{D_1}$, with $\lambda_1 \in \Lambda_1 \cap (V + \overline{D_1}) = \Lambda_1 \cap (A + D)$, are disjoint, have volume 1, and cover V, since, for any $v \in V$, $v + \overline{D_1}$ intersects Λ_1 and $\overline{D_1}$ is 0-symmetric. From this it follows that $\left|\Lambda_1 \cap (V + \overline{D_1})\right| \geq |A|$, which concludes the proof that (24) holds.

So we can construct a one-to-one map $f : \Lambda_0 \to \Lambda_1$, and similarly a one-to-one map $g : \Lambda_1 \to \Lambda_0$, such that

$$|\lambda_0 - f(\lambda_0)| \leq 2\rho \quad \text{and} \quad |\lambda_1 - g(\lambda_1)| \leq 2\rho, \quad (\lambda_0 \in \Lambda_0, \lambda_1 \in \Lambda_1). \tag{25}$$

To get a bijection between the two lattices where each pair of corresponding elements are at most 2ρ distance apart, we essentially repeat the proof of the Bernstein–Schroeder theorem. We partition the (disjoint) union of Λ_0 and Λ_1 into "cycles" of the type

$$\cdots \stackrel{g}{\to} \lambda_0 \stackrel{f}{\to} \lambda_1 \stackrel{g}{\to} \lambda_0' \stackrel{f}{\to} \lambda_1' \stackrel{g}{\to} \cdots \tag{26}$$

Such a cycle can be of finite length (necessarily even), be infinite in both directions or be infinite only to the right. In all these cases one can pair the elements of the two lattices in pairs (λ_0, λ_1), with either $\lambda_1 = f(\lambda_0)$ or $\lambda_0 = g(\lambda_1)$, thereby achieving the required upper bound of 2ρ.

3.3 The support of "soft" multi-lattice tiles

Consider a "soft" multi-lattice tile f. That is, $f \in L^1(\mathbf{R}^d)$ and, for each $i = 0, \ldots, n$, the series

$$\sum_{\lambda \in \Lambda_i} f(x - \lambda) \tag{27}$$

converges absolutely and is constant in $x \in \mathbf{R}^d$. An example of such an f is the function in (4) the diameter of whose support increases linearly in n.

The function in (27) is a constant for all i if and only if $\widehat{f}$ vanishes on the union of the lattices Λ_i^*, $i = 0, \ldots, n$, except possibly at 0. Can this information be used to show that the diameter of the support of such an f admits a lower bound that increases with n? The information about the vanishing set of the Fourier Transform of f has previously been used succsesfuly to prove upper bounds on the rate of decay of Steinhaus sets [**Be**, **K1**, **K2**].

Acknowledgment: I would like to thank Maria Loukaki, Vassilis Nestoridis, Panagiotis Papasoglu and Nikos Tzanakis for very useful discussions concerning the various aspects of the problem. The detailed work and useful suggestions of the anonymous referee, especially those related to §3.2, are gratefully acknowledged.

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