Tiling and spectral properties of near-cubic domains

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June 12, 2002

1 Introduction

Let E be a measurable set in $\mathbb{R}^n$ such that $0 < |E| < \infty$. We will say that E *tiles* $\mathbb{R}^n$ *by translations* if there is a discrete set $T \subset \mathbb{R}^n$ such that, up to sets of measure 0, the sets $E + t : \ t \in T$ are mutually disjoint and $\bigcup_{t\in T}(E + t) = \mathbb{R}^n$. We call any such T a *translation set* for E, and write $E + T = \mathbb{R}^n$. A tiling $E + T = \mathbb{R}^n$ is called *periodic* if it admits a period lattice of rank n; it is a *lattice tiling* if T itself is a lattice. Here and below, a *lattice* in $\mathbb{R}^n$ will always be a set of the form $T\mathbb{Z}^n$, where T is a linear transformation of rank n.

It is known [19], [18] that if a convex set E tiles $\mathbb{R}^n$ by translations, it also admits a lattice tiling. A natural question is whether a similar result holds if E is "sufficiently close" to being convex, e.g. if it is close enough (in an appropriate sense) to a n-dimensional cube. In this paper we prove that this is indeed so in dimensions 1 and 2; we also construct a counterexample in dimensions $n \geq 3$.

A major unresolved problem in the mathematical theory of tilings is the *periodic tiling conjecture*, which asserts that any E which tiles $\mathbb{R}^n$ by translations must also admit a periodic tiling. (See [3] for an overview of this and other related questions.) The conjecture has been proved for all bounded measurable subsets of $\mathbb{R}$ [16], [12] and for topological discs in $\mathbb{R}^2$ [2], [8]. Our Theorem 2 and Corollary 1 prove the conjecture for near-square domains in $\mathbb{R}^2$. We emphasize that no assumptions on the topology of E are needed; in particular, E is not required to be connected and may have infinitely many connected components.

Our work was also motivated in part by a conjecture of Fuglede [1]. We call a set E *spectral* if there is a discrete set $\Lambda \subset \mathbb{R}^n$, which we call a *spectrum* for E, such that $\{e^{2\pi i\lambda\cdot x} : \lambda \in \Lambda\}$ is an orthogonal basis for $L^2(E)$. Fuglede conjectured that E is spectral if and only if it tiles $\mathbb{R}^n$ by translations, and proved it under the assumption that either the translation set T or the spectrum Λ is a lattice. This problem was addressed in many recent papers (see e.g. [4], [7], [10], [13], [14], [15], [16], [17]), and in particular the conjecture has been proved for convex regions in $\mathbb{R}^2$ [9], [5], [6].

It follows from our Theorem 1 and from Fuglede's theorem that the conjecture is true for $E \subset \mathbb{R}$ such that E is contained in an interval of length strictly less than $3|E|/2$. (This was proved in [15] in the special case when E is a union of finitely many intervals of equal length.) In dimension 2, we obtain the "tiling $\Rightarrow$ spectrum" part of the conjecture for near-square domains. Namely, if $E \subset \mathbb{R}^2$ tiles $\mathbb{R}^2$ and satisfies the assumptions of Theorem 2 or Corollary 1, it also admits a lattice tiling, hence it is a spectral set by Fuglede's theorem on the lattice case of his conjecture. We do not know how to prove the converse implication.

Our main results are the following.

Theorem 1 *Suppose $E \subseteq [0, L]$ is measurable with measure 1 and $L = 3/2 - \epsilon$ for some $\epsilon > 0$. Let $\Lambda \subset \mathbb{R}$ be a discrete set containing 0. Then*
(a) if $E + \Lambda = \mathbb{R}$ is a tiling, it follows that $\Lambda = \mathbb{Z}$.
(b) if Λ is a spectrum of E, it follows that $\Lambda = \mathbb{Z}$.

The upper bound $L < 3/2$ in Theorem 1 is optimal: the set $[0, 1/2] \cup [1, 3/2]$ is contained in an interval of length 3/2, tiles $\mathbb{Z}$ with the translation set $\{0, 1/2\} + 2\mathbb{Z}$, and has the spectrum $\{0, 1/2\} + 2\mathbb{Z}$, but does not have either a lattice translation set or a lattice spectrum. This example has been known to many authors; an explicit calculation of the spectrum is given e.g. in [14].

Theorem 2 *Let $E \subset \mathbb{R}^2$ be a measurable set such that $[0, 1]^2 \subset E \subset [-\epsilon, 1 + \epsilon]^2$ for $\epsilon > 0$ small enough. Assume that E tiles $\mathbb{R}^2$ by translations. Then E also admits a tiling with a lattice $\Lambda \subset \mathbb{R}^2$ as the translation set.*

Our proof works for $\epsilon < 1/33$; we do not know what is the optimal upper bound for ϵ.

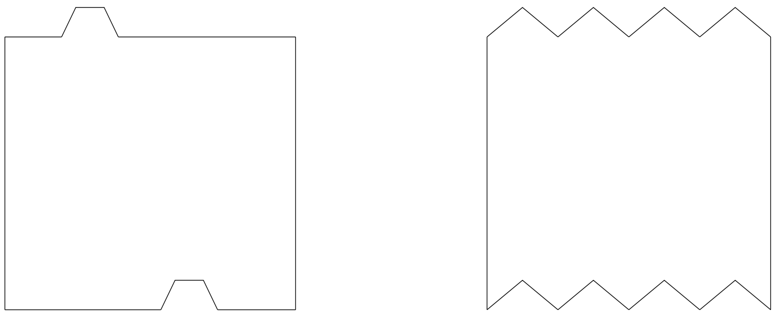

Figure 1: Examples of near-square regions which tile $\mathbb{R}^2$. Note that the second region also admits aperiodic (hence non-lattice) tilings.

Corollary 1 *Let $E \subset \mathbb{R}^2$ be a measurable set such that $|E| = 1$ and E is contained in a square of sidelength $1 + \epsilon$ for $\epsilon > 0$ small enough. If E tiles $\mathbb{R}^2$ by translations, then it also admits a lattice tiling.*

Theorem 3 *Let $n \geq 3$. Then for any $\epsilon > 0$ there is a set $E \subset \mathbb{R}^n$ with $[0, 1]^n \subset E \subset [-\epsilon, 1 + \epsilon]^n$ such that E tiles $\mathbb{R}^n$ by translations, but does not admit a lattice tiling.*

2 The one-dimensional case

In this section we prove Theorem 1. We shall need the following crucial lemma.

Lemma 1 *Suppose that $E \subseteq [0, L]$ is measurable with measure 1 and that $L = 3/2 - \epsilon$ for some $\epsilon > 0$. Then*

$$|E \cap (E + x)| > 0 \quad \textit{whenever } 0 \leq x < 1. \tag{1}$$

Proof of Lemma 1. We distinguish the cases (i) $0 \le x \le 1/2$, (ii) $1/2 < x \le 3/4$ and (iii) $3/4 < x < 1$.

(i) $0 \le x \le 1/2$

This is the easy case as $E \cup (E+x) \subseteq [0, L+1/2] = [0, 2-\epsilon]$. Since this interval has length less than 2, the sets E and $E+x$ must intersect in positive measure.

(ii) $1/2 < x \le 3/4$

Let $x = 1/2 + \alpha$, $0 < \alpha \le 1/4$. Suppose that $|E \cap (E+x)| = 0$. Then $1 + 2\alpha \le 3/2$ and

$$|(E \cap [0, x]) \cup (E \cap [x, 2x])| \le x,$$

as the second set does not intersect the first when shifted back by x. This implies that

$$|E| \le x + (3/2 - \epsilon - 2x) = 3/2 - \epsilon - x = 1 - \epsilon - \alpha < 1,$$

a contradiction as $|E| = 1$.

(iii) $3/4 \le x < 1$

Let $x = 3/4 + \alpha$, $0 < \alpha < 1/4$. Suppose that $|E \cap (E+x)| = 0$. Then

$$|(E \cap [0, 3/4 - \alpha - \epsilon]) \cup (E \cap [3/4 + \alpha, 3/2 - \epsilon])| \le 3/4 - \alpha - \epsilon,$$

for the second set translated to the left by x does not intersect the first. This implies that

$$|E| \le (3/4 - \alpha - \epsilon) + 2\alpha + \epsilon = 3/4 + \alpha < 1,$$

a contradiction.
□

We need to introduce some terminology. If f is a nonnegative integrable function on $\mathbb{R}^d$ and Λ is a subset of $\mathbb{R}^d$, we say that $f + \Lambda$ is a packing if, almost everywhere,

$$\sum_{\lambda \in \Lambda} f(x - \lambda) \le 1. \qquad (2)$$

We say that $f + \Lambda$ is a tiling if equality holds almost everywhere. When $f = \chi_E$ is the indicator function of a measurable set, this definition coincides with the classical geometric notions of packing and tiling.

We shall need the following theorem from [10].

Theorem 4 *If $f, g \ge 0$, $\int f(x)dx = \int g(x)dx = 1$ and both $f + \Lambda$ and $g + \Lambda$ are packings of $\mathbb{R}^d$, then $f + \Lambda$ is a tiling if and only if $g + \Lambda$ is a tiling.*

Proof of Theorem 1. (a) Suppose $E + \Lambda$ is a tiling. From Lemma 1 it follows that any two elements of Λ differ by at least 1. This implies that $\chi_{[0,1]} + \Lambda$ is a packing, hence it is also a tiling by Theorem 4. Since $0 \in \Lambda$, we have $\Lambda = \mathbb{Z}$.

(b) Suppose that Λ is a spectrum of E. Write

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$$

for the measure of one unit mass at each point of Λ. Our assumption that Λ is a spectrum for E implies that

$$|\widehat{\chi_E}|^2 + \Lambda = \mathbb{R}$$

is a tiling (see, for example, [10]). This, in turn, implies that $\mathrm{dens}\,\Lambda = 1$.

We now use the following result from [10]:

Theorem 5 *Suppose that $f \ge 0$ is not identically 0, that $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \ge 0$ has compact support and $\Lambda \subset \mathbb{R}^d$. If $f + \Lambda$ is a tiling then*

$$\mathrm{supp}\,\widehat{\delta_\Lambda} \subseteq \left\{\widehat{f} = 0\right\} \cup \{0\}. \tag{3}$$

Let us emphasize here that the object $\widehat{\delta_\Lambda}$, the Fourier Transform of the tempered measure δ_Λ, is in general a tempered distribution and need not be a measure.

For $f = |\widehat{\chi_E}|^2$ Theorem 5 implies

$$\mathrm{supp}\,\widehat{\delta_\Lambda} \subseteq \{0\} \cup \{\chi_E * \widetilde{\chi_E} = 0\}, \tag{4}$$

since $\chi_E * \widetilde{\chi_E}$ is the Fourier transform of $|\widehat{\chi_E}|^2$ (where $\widetilde{g}(x) = \overline{g(-x)}$). But

$$\{\chi_E * \widetilde{\chi_E} = 0\} = \{x : \ |E \cap (E + x)| = 0\}.$$

This and Lemma 1 imply that

$$\mathrm{supp}\,\widehat{\delta_\Lambda} \cap (-1, 1) = \{0\}.$$

Let

$$K_\delta(x) = \max\{0, 1 - (1+\delta)|x|\} = (1+\delta)\chi_{I_\delta} * \widetilde{\chi_{I_\delta}}(x),$$

where $I_\delta = [0, \frac{1}{1+\delta}]$, be a Fejér kernel (we will later take $\delta \to 0$). Then $\widehat{K_\delta} = (1+\delta)|\widehat{\chi_{I_\delta}}|^2$ is a non-negative continuous function and, after calculating $\widehat{\chi_{I_\delta}}$, it follows that

$$\widehat{K_\delta}(0) = \frac{1}{1+\delta}$$

and

$$\left\{x : \widehat{K_\delta}(x) = 0\right\} = (1+\delta)(\mathbb{Z} \setminus \{0\}). \tag{5}$$

Next, we use the following result from [11]:

Theorem 6 *Suppose that $\Lambda \in \mathbb{R}^d$ is a multiset with density ρ, $\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda$, and that $\widehat{\delta_\Lambda}$ is a measure in a neighborhood of 0. Then $\widehat{\delta_\Lambda}(\{0\}) = \rho$.*

Remark. The proof of Theorem 6 shows that the assumption of $\widehat{\delta_\Lambda}$ being a measure in a neighborhood of zero is superfluous, if one knows a priori that $\widehat{\delta_\Lambda}$ is supported only at zero, in a neighborhood of zero. Indeed, what is shown in that proof is that, as $t \to \infty$, the quantity $\widehat{\delta_\Lambda}(\phi(tx))$ remains bounded, for any C_c^∞ test function ϕ. If $\widehat{\delta_\Lambda}$ were not a measure near 0 but had support only at 0, locally, this quantity would grow like a polynomial in t of degree equal to the degree of the distribution at 0.

Applying Theorem 6 and the remark following it we obtain that $\widehat{\delta_\Lambda}$ is equal to δ_0 in a neighborhood of 0, since Λ has density 1.

Next, we claim that

$$\sum_{\lambda\in\Lambda} \widehat{K_\delta}(x-\lambda) = 1, \quad \text{for a.e. } x.$$

Indeed, take ψ_ϵ to be a smooth, positive-definite approximate identity, supported in $(-\epsilon,\epsilon)$, and take $\epsilon=\epsilon(\delta)$ to be small enough so that $\operatorname{supp}\psi_\epsilon * K_\delta \subset (-1,1)$. We have then

$$\begin{aligned}
\sum_{\lambda\in\Lambda} \widehat{K_\delta}(x-\lambda) &= \lim_{\epsilon\to 0} \sum_{\lambda\in\Lambda} \widehat{\psi_\epsilon}(x-\lambda)\widehat{K_\delta}(x-\lambda) \\
&= \lim_{\epsilon\to 0} \delta_\Lambda\left((\widehat{\psi_\epsilon}\widehat{K_\delta})(x)\right) \\
&= \lim_{\epsilon\to 0} \widehat{\delta_\Lambda}\left((\psi_\epsilon * K_\delta)(x)\right) \\
&= \lim_{\epsilon\to 0} \delta_0\left((\psi_\epsilon * K_\delta)(x)\right) \quad \text{(for } \epsilon \text{ small enough)} \\
&= \lim_{\epsilon\to 0} \psi_\epsilon * K_\delta(0) \\
&= K_\delta(0) \\
&= 1,
\end{aligned}$$

which establishes the claim. Applying this for $x\to 0$ and isolating the term $\lambda=0$ we get

$$1 = \frac{1}{1+\delta} + \sum_{0\neq\lambda\in\Lambda} \widehat{K_\delta}(-\lambda).$$

Letting $\delta\to 0$ we obtain that $\widehat{K_\delta}(-\lambda)\to 0$ for each $\lambda\in\Lambda\setminus\{0\}$, which implies that each such λ is an integer, as $\mathbb{Z}\setminus\{0\}$ is the limiting set of the zeros of $\widehat{K_\delta}$.

To get that $\Lambda=\mathbb{Z}$ notice that $\chi_{[0,1]}+\Lambda$ is a packing. By Theorem 4 again we get that $\chi_{[0,1]}+\Lambda$ is in fact a tiling, hence $\Lambda=\mathbb{Z}$.
□

3 Planar regions

Proof of Theorem 2. We denote the coordinates in $\mathbb{R}^2$ by (x_1,x_2). For $0\le a\le b\le 1$ we will denote

$$\begin{aligned}
E_1(a,b) &= (E\cap\{a\le x_1\le b,\ x_2\le 0\})\cup\{a\le x_1\le b,\ x_2\ge 0\}, \\
E_2(a,b) &= (E\cap\{a\le x_1\le b,\ x_2\ge 0\})\cup\{a\le x_1\le b,\ x_2\le 0\}, \\
F_1(a,b) &= (E\cap\{a\le x_2\le b,\ x_1\le 0\})\cup\{a\le x_2\le b,\ x_1\ge 0\}, \\
F_2(a,b) &= (E\cap\{a\le x_2\le b,\ x_1\ge 0\})\cup\{a\le x_2\le b,\ x_1\le 0\}.
\end{aligned}$$

We will also use $S_{a,b}$ to denote the vertical strip $[a,b]\times\mathbb{R}$. Let $v=(v_1,v_2)\in\mathbb{R}^2$. We will say that $E_2(a,b)$ *complements* $E_1(a',b')+v$ if $E_1(a',b')+v$ is positioned above $E_2(a,b)$ so that (up to sets of measure 0) the two sets are disjoint and their union is $S_{a,b}$. In particular, we must have $a'+v_1=a$ and $b'+v_1=b$. We will write $\widetilde{E}_1(a,b)=S_{a,b}\setminus E_1(a,b)$, and similarly for E_2. Finally, we write $A\sim B$ if the sets A and B are equal up to sets of measure 0.

Lemma 2 *Let $0 < s'' < s' < s < 2s''$. Suppose that $E_1(a, a+s)+v$, $E_1(a, a+s')+v'$, $E_1(a, a+s'')+v''$ complement $E_2(b-s, b)$, $E_2(b-s', b)$, $E_2(b-s'', b)$ respectively. Then the points v, v', v'' are collinear. Moreover, the absolute value of the slope of the line through v, v'' is bounded by $\epsilon(2s''-s)^{-1}$.*

Applying the lemma to the symmetric reflection of E about the line $x_2 = 1/2$, we find that the conclusions of the lemma also hold if we assume that $E_2(a, a+s)+v$, $E_2(a, a+s')+v'$, $E_2(a, a+s'')+v''$ complement $E_1(b-s, b)$, $E_1(b-s', b)$, $E_1(b-s'', b)$ respectively. Furthermore, we may interchange the x_1 and x_2 coordinates and obtain the analogue of the lemma with E_1, E_2 replaced by F_1, F_2.

Proof of Lemma 2. Let $v = (v_1, v_2)$, $v' = (v_1', v_2')$, $v'' = (v_1'', v_2'')$. We first observe that if $v_1 = v_1''$, it follows from the assumptions that $v = v''$ and there is nothing to prove. We may therefore assume that $v_1 \neq v_1''$. We do, however, allow $v' = v$ or $v' = v''$.

It follows from the assumptions that $E_2(b-s'', b)$ complements each of $E_1(a, a+s'')+v''$, $E_1(a+s'-s'', a+s')+v'$, $E_1(a+s-s'', a+s)+v$. Hence

$$E_1(a+s'-s'', a+s') \sim E_1(a, a+s'')+(v''-v'),$$

$$E_1(a+s-s'', a+s) \sim E_1(a, a+s'')+(v''-v).$$

Let n be the unit vector perpendicular to $v-v''$ and such that $n_2 > 0$. For $t \in \mathbb{R}$, let $P_t = \{x : \ x \cdot n \leq t\}$. We define for $0 \leq c \leq c' \leq 1$:

$$\alpha_{c,c'} = \inf\{t \in \mathbb{R} : \ |E_1(c, c') \cap P_t| > 0\},$$

$$\beta_{c,c'} = \sup\{t \in \mathbb{R} : \ |\widetilde{E}_1(c, c') \setminus P_t| > 0\}.$$

We will say that x is a *low point* of $E_1(c, c')$ if $x \in S_{c,c'}$, $x \cdot n = \alpha_{c,c'}$, and for any open disc D centered at x we have

$$|D \cap E_1(c, c')| > 0. \tag{6}$$

Similarly, we call y a *high point* of $\widetilde{E}_1(c, c')$ if $y \in S_{c,c'}$, $y \cdot n = \beta_{c,c'}$, and for any open disc D centered at y we have

$$|D \cap \widetilde{E}_1(c, c')| > 0. \tag{7}$$

It is easy to see that such points x, y actually exist. Indeed, by the definition of $\alpha_{c,c'}$ and an obvious covering argument, for any $\alpha > \alpha_{c,c'}$ there are points x' such that $x' \cdot n \leq \alpha$ and that (6) holds for any disc D centered at x'. Thus the set of such points x' has at least one accumulation point x on the line $x \cdot n = \alpha_{c,c'}$. It follows that any such x is a low point of $E_1(c, c')$. The same argument works for y.

The low and high points need not be unique; however, all low points x of $E_1(c, c')$ lie on the same line $x \cdot n = \alpha_{c,c'}$ parallel to the vector $v - v''$, and similarly for high points. Furthermore, the low and high points of $E_1(c, c')$ do not change if $E_1(c, c')$ is modified by a set of measure 0.

Let now $A = E_1(a, a+s'')$, and let x be a low point of A. Since $s < 2s''$, we have

$$B := E_1(a, a+s) = E_1(a, a+s'') \cup E_1(a+s-s'', a+s) \sim A \cup (A+v''-v),$$

hence x is also a low point of B with respect to $v - v''$. Now note that

$$E_1(a+s'-s'', a+s') \sim A+(v''-v')$$

intersects any open neighbourhood of $x+(v''-v')$ in positive measure. But on the other hand, $E_1(a+s'-s'', a+s') \subset B$. By the extremality of x in B, $x+(v''-v')$ lies on or above the line segment joining x and $x+(v''-v)$, hence $v''-v'$ lies on or above the line segment joining 0 and $v''-v$.

Repeating the argument in the last paragraph with x replaced by a high point y of $\widetilde{E}_1(a, a+s'')$, we obtain that $v''-v'$ lies on or below the line segment joining 0 and $v''-v$. Hence v, v', v'' are collinear.

Finally, we estimate the slope of the line through v, v''. We have to prove that

$$\frac{2s''-s}{s-s''}|v_2''-v_2| \leq \epsilon \tag{8}$$

(recall that $v_1''-v_1 = s-s''$). Define x as above, and let $k \in \mathbb{Z}$. Iterating translations by $v-v''$ (in both directions), we find that $x+k(v-v'')$ is a low point of B as long as it belongs to B, i.e. as long as

$$a \leq x_1 + k(s-s'') \leq a+s.$$

The number of such k's is at least $\frac{s}{s-s''}-1$. On the other hand, all low points of B lie in the rectangle $a \leq x_1 \leq a+s, -\epsilon \leq x_2 \leq 0$. Hence

$$(\frac{s}{s-s''}-2)|v_2''-v_2| \leq \epsilon,$$

which is (8).
□

We return to the proof of Theorem 2. Since E is almost a square, we know roughly how the translates of E can fit together. Locally, any tiling by E is essentially a tiling by a "solid" 1×1 square with "margins" of width between 0 and 2ϵ (see Fig. 2).

We first locate a "corner". Namely, we may assume that the tiling contains E and its translates $E+u$, $E+v$, where

$$1 \leq u_1 \leq 1+2\epsilon,\ -2\epsilon \leq u_2 \leq 2\epsilon, \tag{9}$$

$$0 \leq v_1 \leq \frac{1}{2}+\epsilon,\ 1 \leq v_2 \leq 1+2\epsilon. \tag{10}$$

This can always be achieved by translating the tiled plane and taking symmetric reflections of it if necessary.

Let $E+w$ be the translate of E which fits into this corner:

$$v_1+1 \leq w_1 \leq v_1+1+2\epsilon,\ u_2+1 \leq w_2 \leq u_2+1+2\epsilon. \tag{11}$$

We will prove that $w = u+v$ (without the ϵ-errors).

From (11), (9), (10) we have

$$1 \leq w_1 \leq \frac{3}{2}+3\epsilon,\ -4\epsilon \leq w_2 - v_2 \leq 4\epsilon.$$

Hence w satisfies both of the following.

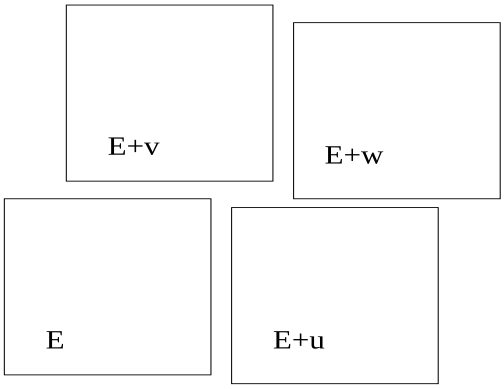

Figure 2: A "corner" and a fourth near-square.

(A) $E_2(0, 1 - (w_1 - u_1))$ complements $E_1(w_1 - u_1, 1) + (w - u)$, and

$$1 - (w_1 - u_1) = 1 - w_1 + u_1 \geq 1 + 1 - (\frac{3}{2} + 3\epsilon) = \frac{1}{2} - 3\epsilon,$$

$$|(w_1 - u_1) - v_1| = |(w_1 - v_1) - u_1| \leq 2\epsilon.$$

(B) $-4\epsilon \leq w_2 - v_2 \leq 4\epsilon$, and $F_2(r, t)$ complements $F_1(r', t') + (w - v)$, where

$$r = \max(0, w_2 - v_2),\ r' = \max(0, v_2 - w_2),$$

$$t = 1 - \max(0, v_2 - w_2),\ t' = 1 - \max(0, w_2 - v_2).$$

If $w = u + v$, we have $w - u = v$, $w - v = u$, hence by considering the "corner" $E, E + u, E + v$ we see that both (A) and (B) hold. Assuming that ϵ is small enough, we shall prove that:

1. All points w satisfying (A) lie on a fixed straight line l_1 making an angle less than $\pi/4$ with the x_1 axis.

2. All points w satisfying (B) lie on a fixed straight line l_2 making an angle at most $\pi/4$ with the x_2 axis.

It follows that there can be at most one w which satisfies both (A) and (B), since l_1 and l_2 intersect only at one point. Consequently, if $E + w$ is the translate of E chosen as above, we must have $w = u + v$. Now it is easy to see that $E + \Lambda$ is a tiling, where Λ is the lattice $\{ku + mv :\ k, m \in \mathbb{Z}\}$.

We first prove 1. Suppose that $w, w', w'', \ldots$ (not necessarily all distinct) satisfy (A). By the assumptions in (A), we may apply Lemma 2 with E_1 and E_2 interchanged and with $a = 0$, $b = 1$, $s = 1 - (w_1 - u_1), s' = 1 - (w'_1 - u_1), \ldots \geq \frac{1}{2} - 3\epsilon$. From the second inequality in (A) and the triangle inequality we also have $|s - s''| \leq 4\epsilon$. We find that all w satisfying (A) lie on a line l_1 with slope bounded by

$$\frac{\epsilon}{|2s'' - s|} \leq \frac{\epsilon}{s'' - |s'' - s|} \leq \frac{\epsilon}{1/2 - 7\epsilon},$$

which is less than 1 if $\epsilon < 1/16$.

To prove 2., we let w, w', w'' be three (not necessarily distinct) points satisfying (B) and such that $w_2 \leq w_2' \leq w_2''$. We then apply the obvious analogue of Lemma 2 with E_1, E_2 replaced by F_1, F_2 and with $a = \max(v_2 - w_2, 0) \leq 4\epsilon$, $b = 1 - \max(v_2 - w_2) \geq 1 - 4\epsilon$. From the estimates in (B) we have $1 - 16\epsilon \leq s, s', s'' \leq 1$, hence $|2s'' - s| \geq 2 - 32\epsilon - 1 = 1 - 32\epsilon$. We conclude that all w satisfying (B) lie on a line l_2 such that the inverse of the absolute value of its slope is bounded by $\frac{\epsilon}{1-32\epsilon}$. This is at most 1 if $\epsilon \leq 1/33$.
□

Proof of Corollary 1. Let $Q = [0,1] \times [0,1]$. By rescaling, it suffices to prove that for any $\epsilon > 0$ there is a $\delta > 0$ such that if $E \subset Q$, E tiles $\mathbb{R}^2$ by translations, and $|E| \geq 1 - \delta$, then E contains the square

$$Q_\epsilon = [\epsilon, 1-\epsilon] \times [\epsilon, 1-\epsilon]$$

(up to sets of measure 0). The result then follows from Theorem 2.

Let E be as above, and suppose that $Q_\epsilon \setminus E$ has positive measure. Since E tiles $\mathbb{R}^2$, there is a $v \in \mathbb{R}^2$ such that $|E \cap (E+v)| = 0$ and $|Q_\epsilon \cap (E+v)| > 0$. We then have

$$|E \cup (E+v)| = |E| + |E+v| \geq 2 - 2\delta,$$

but also

$$|E \cup (E+v)| \leq |Q \cup (Q+v)| \leq 2 - \epsilon^2,$$

since $E \subset Q$, $E + v \subset Q + v$, and $Q_\epsilon \cap (Q+v) \neq \emptyset$ so that $|Q \cap (Q+v)| \geq \epsilon^2$. This is a contradiction if δ is small enough.
□

4 A counterexample in higher dimensions

In this section we prove Theorem 3. It suffices to construct E for $n = 3$, since then $E \times [0,1]^{n-3}$ is a subset of $\mathbb{R}^n$ with the required properties.

Let (x_1, x_2, x_3) denote the Cartesian coordinates in $\mathbb{R}^3$. It will be convenient to rescale E so that $[\epsilon, 1]^3 \subset E \subset [0, 1+\epsilon]^3$.

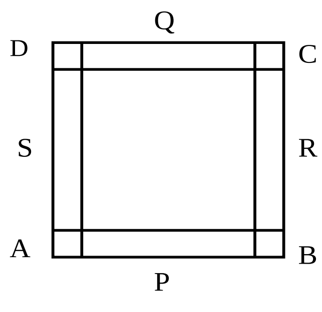

Figure 3: The construction of E.

We construct E as follows. We let E be bounded from below and above by the planes $x_3 = 0$ and $x_3 = 1$ respectively. The planes $x_1 = \epsilon, x_1 = 1, x_2 = \epsilon, x_2 = 1$ divide the cube $[0, 1+\epsilon]^3$ into

9 parts (Figure 3). The middle part is entirely contained in E. We label by A, B, C, D, P, Q, R, S the remaining 8 segments as shown in Figure 3. We then let

$$E \cap P = P \cap \left\{0 \le x_3 \le \frac{1}{8} \text{ or } \frac{1}{2} \le x_3 \le \frac{5}{8}\right\},$$

$$E \cap R = R \cap \left\{0 \le x_3 \le \frac{1}{8} \text{ or } \frac{1}{2} \le x_3 \le \frac{5}{8}\right\},$$

$$E \cap Q = Q \cap \left\{0 \le x_3 \le \frac{1}{4} \text{ or } \frac{3}{8} \le x_3 \le \frac{3}{4} \text{ or } \frac{7}{8} \le x_3 \le 1\right\},$$

$$E \cap S = S \cap \left\{0 \le x_3 \le \frac{1}{4} \text{ or } \frac{3}{8} \le x_3 \le \frac{3}{4} \text{ or } \frac{7}{8} \le x_3 \le 1\right\},$$

and

$$E \cap A = A \cap \left\{0 \le x_3 \le \frac{1}{16}\right\}, \; E \cap C = A \cap \left\{\frac{1}{2} \le x_3 \le \frac{9}{16}\right\},$$

$$E \cap B = B \cap \left\{\frac{5}{16} \le x_3 \le \frac{3}{4}\right\}, \; E \cap D = D \cap \left\{0 \le x_3 \le \frac{1}{4} \text{ or } \frac{13}{16} \le x_3 \le 1\right\}.$$

We also denote $K = \bigcup_{j \in \mathbb{Z}} (E + (0, 0, j))$.

Let $E + T$ be a tiling of $\mathbb{R}^3$, and assume that $0 \in T$. Suppose that $E + v$ and $E + w$ are neighbours in this tiling so that the vertical sides of $(E \cap P) + v$ and $(E \cap Q) + w$ meet in a set of non-zero two-dimensional measure. Then we must have $v - w = (0, 1, (v - w)_3)$, where $(v - w)_3 \in \{\pm\frac{1}{4}, \pm\frac{3}{4}\}$. A similar statement holds with P, Q replaced by R, S and with the x_1, x_2 coordinates interchanged. We deduce that the tiling consists of copies of E stacked into identical vertical "columns" $K_{ij} = K + (i, j, t_{ij})$, arranged in a rectangular grid in the $x_1 x_2$ plane and shifted vertically so that $t_{i+1,j} - t_{ij}$ and $t_{i,j+1} - t_{ij}$ are always $\pm\frac{1}{4}$. We will use matrices (t_{ij}) to encode such a tiling or portions thereof.

It is easy to see that (t_{ij}), where $t_{ij} = 0$ if $i + j$ is even and $\frac{1}{4}$ if $i + j$ is odd, is indeed a tiling. It remains to show that E does not admit a lattice tiling. Indeed, the four possible choices of the generating vectors in any lattice (t_{ij}) with $t_{ij} = \pm\frac{1}{4}$ produce the configurations

$$\begin{pmatrix} 0 & t \\ t & 2t \end{pmatrix}, \begin{pmatrix} 2t & t \\ t & 0 \end{pmatrix}, \begin{pmatrix} 0 & t \\ -t & 0 \end{pmatrix}, \begin{pmatrix} 0 & -t \\ t & 0 \end{pmatrix}.$$

But it is easy to see that the corners A, B, C, D do not match if so translated.

References

[1] B. Fuglede: *Commuting self-adjoint partial differential operators and a group-theoretic problem*, J. Funct. Anal. 16 (1974), 101–121

[2] D. Girault-Beauquier, M. Nivat: *Tiling the plane with one tile*, in: *Topology and Category Theory in Computer Science*, G.M. Reed, A.W. Roscoe, R.F. Wachter (eds.), Oxford Univ. Press 1989, 291–333.

[3] B. Grünbaum, G.C. Shepard: *Tilings and patterns*, New York: Freeman 1987.

[4] A. Iosevich, N. H. Katz, S. Pedersen: *Fourier bases and a distance problem of Erdös*, Math. Res. Letters 6 (1999), 251–255.

[5] A. Iosevich, N. H. Katz, T. Tao: *Convex bodies with a point of curvature do not have Fourier bases*, Amer. J. Math. 123 (2001), 115–120.

[6] A. Iosevich, N.H. Katz, T.Tao: *Fuglede conjecture holds for convex planar domains*, preprint, 2001.

[7] P. Jorgensen, S. Pedersen: *Spectral pairs in Cartesian coordinates*, J. Fourier Anal. Appl. 5 (1999), 285–302.

[8] R. Kenyon: *Rigidity of planar tilings*, Invent. Math. 107 (1992), 637–651.

[9] M. Kolountzakis: *Non-symmetric convex domains have no basis of exponentials*, Illinois J. Math. 44 (2000), 542–550.

[10] M.N. Kolountzakis: *Packing, Tiling, Orthogonality and Completeness,* Bull. L.M.S. **32** (2000), 5, 589–599.

[11] M.N. Kolountzakis: *On the structure of multiple translational tilings by polygonal regions*, Discrete Comput. Geom. **23** (2000), 4, 537–553.

[12] M.N. Kolountzakis, J.C. Lagarias: *Structure of tilings of the line by a function*, Duke Math. J. 82 (1996), 653–678.

[13] M.N. Kolountzakis, M. Papadimitrakis: *A class of non-convex polytopes that admit no orthonormal basis of exponentials*, preprint, 2001.

[14] I. Łaba: *Fuglede's conjecture for a union of two intervals*, Proc. AMS 121 (2001), 2965–2972.

[15] I. Łaba: *The spectral set conjecture and multiplicative properties of roots of polynomials*, J. London Math. Soc., to appear.

[16] J. Lagarias, Y. Wang: *Tiling the line with translates of one tile*, Inv. Math. 124 (1996), 341–365.

[17] J. Lagarias, Y. Wang: *Spectral sets and factorization of finite abelian groups*, J. Funct. Anal. 73 (1997), 122–134.

[18] P. McMullen: *Convex bodies which tile the space by translation*, Mathematika 27 (1980), 113–121.

[19] B.A. Venkov: *On a class of Euclidean polyhedra*, Vestnik Leningrad Univ. Ser. Mat. Fiz. Him. 9 (1954), 11–31.

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