DISCREPANCY OF LINE SEGMENTS FOR GENERAL LATTICE CHECKERBOARDS

MIHAIL N. KOLOUNTZAKIS

Abstract. In a series of papers recently "checkerboard discrepancy" has been introduced, where a black-and-white checkerboard background induces a coloring on any curve, and thus a discrepancy, i.e., the difference of the length of the curve colored white and the length colored black. Mainly straight lines and circles have been studied and the general situation is that, no matter what the background coloring, there is always a curve in the family studied whose discrepancy is at least of the order of the square root of the length of the curve.

In this paper we generalize the shape of the background, keeping the lattice structure. Our background now consists of lattice copies of any bounded fundamental domain of the lattice, and not necessarily of squares, as was the case in the previous papers. As the decay properties of the Fourier Transform of the indicator function of the square were strongly used before, we now have to use a quite different proof, in which the tiling and spectral properties of the fundamental domain play a role.

Contents

1. Introduction

1.1. Checkerboard discrepancies. In the series of papers [5, 7, 8] the discrepancies of curves against a two-colored checkerboard background were introduced and upper and lower bounds for straight lines and circles were given. Briefly, the discrepancy of a curve Γ against a

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two-colored background, say a black-and-white background, is the absolute difference between the lengths of the black and the white part of Γ. See Figure 1.

The motivating question was the following.

> Can we color the usual lattice subdivision of the plane into a checkerboard using two colors so that any line segment placed on the plane has *bounded* discrepancy?

It was shown in [7] that the answer is negative: for any two-coloring of the infinite checkerboard, i.e., of the partition

$$\mathbb{R}^2 = \bigcup_{k \in \mathbb{Z}^2} \left(k + [0,1)^2\right),$$

there are arbitrarily large line segments I whose discrepancy is at least $C\sqrt{|I|}$, where $|I|$ is the length of the segment I and C is an absolute constant. Further questions, such as constructions of colorings that give as small discrepancy as possible, as well as the discrepancy of other shapes, mainly circles and circular arcs, have been dealt with in [5,7,8].

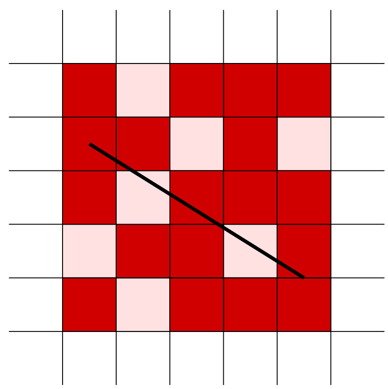

FIGURE 1. A two-colored checkerboard with a line segment on it.

1.2. **Geometric discrepancy.** The question dealt with in this paper as well as in [5,7,8] falls naturally into the subject of geometric discrepancy [1–4,9]. In this research area there is usually an underlying measure μ as well as a family $\mathcal{F}$ of allowed subsets of Euclidean space, on which the measure μ is evaluated and upper and lower bounds are sought on the range of μ on $\mathcal{F}$. The most classical case is that where μ is a normalized collection of points masses in the unit square minus Lebesgue measure and $\mathcal{F}$ consists of all axis-aligned rectangles in the unit square. Usually the underlying measure μ has an atomic part (point masses) and the family $\mathcal{F}$ consists of "fat" sets. In the problem we are studying here the measure μ has no atomic part (it is absolutely continuous) and the collection $\mathcal{F}$ consists of all straight line segments, or other curves, which may be considered thin sets, and, strictly speaking, μ is 0 on these sets. The measure μ is however regular enough to induce a measure on the one-dimensional objects whose discrepancy we study, and it is of this induced measure that we study the range.

We note that discrepancies with respect to non-atomic colorings have been considered by Rogers in [10], [11] and [12] where the author considers, among other things, the discrepancy of lines and half spaces with respect to finite colorings of the plane. Rogers proves lower bounds for the discrepancy of these families of sets with respect to generalized colorings. His results do not seem to be comparable to the results in this paper or those in [5,7,8].

1.3. **New results.** In this paper we generalize the results of [7] to more general two-colorings of the plane. We keep the lattice structure of the original problem but now we tile the plane with much more general shapes than squares. For a lattice $T \subseteq \mathbb{R}^2$ (discrete, additive subgroup of $\mathbb{R}^2$ whose linear span is the plane) we consider a fundamental domain Q of T, i.e., a set containing exactly one element from each coset of T in $\mathbb{R}^2$. This is the same as asking that

$$\mathbb{R}^2 = \bigcup_{t\in T}(t+Q) \quad \text{is a partition of } \mathbb{R}^2.$$

We then color each set $t+Q$ in this partition using using one of two possible colors and this defines the discrepancy of every line segment on the plane on which the coloring function is measurable. This happens for almost all (Lebesgue) straight lines normal to any given direction by Fubini's theorem.

See Figures 2 and 3 for two examples of such generalized checkerboards.

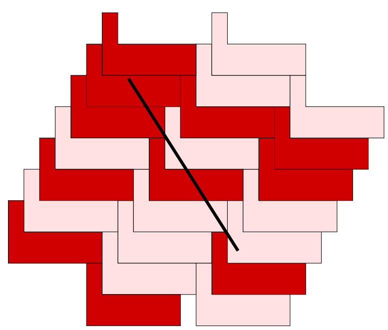

FIGURE 2. A two-colored L-shaped checkerboard.

The following Theorem 1 and Corollary 1 are our main results. The method generalizes to the results in [5,8] as well, but we leave them out to keep the exposition simple.

Theorem 1. *Suppose $T \subseteq \mathbb{R}^2$ is a lattice of area 1 and let Q be a bounded, Lebesgue measurable fundamental domain of T.*

Let also $G \subseteq T$ be a finite subset of lattice points and let the function $f : \mathbb{R}^2 \to \mathbb{C}$ (the coloring*) be defined by*

$$f(x) = \sum_{g\in G} z_g \chi_Q(x-g),$$

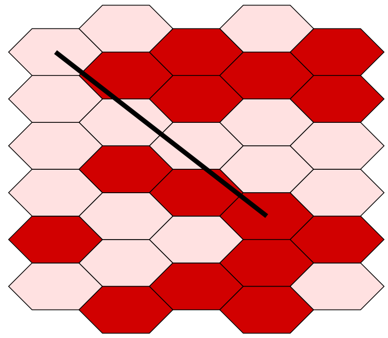

FIGURE 3. A two-colored hexagon-shaped checkerboard.

for some complex numbers z_g, $g \in G$.

Then there is a straight line S *such that* f *is measurable on* S *and*

$$\left|\int_S f\right| \ge C(\operatorname{diam} G)^{-1/2}\left(\sum_{g\in G}|z_g|^2\right)^{1/2},$$

where C *is a positive constant that may depend on* T *and* Q *only.*

Notation: The letter C will always denote a positive number (whose possible dependence on parameters will be clearly stated) which is not the same in all its occurences.

The following speaks more directly about two-colorings of the plane.

Corollary 1. *Suppose* T *and* Q *satisfy the assumptions of Theorem 1. Then there is a positive constant* C*, which may depend on* Q *and* T *only, such that for any assignment* $z_t = \pm 1$*, for* $t \in T$*, and corresponding coloring function*

$$f(x) = \sum_{t\in T} z_g \chi_Q(x-t)$$

there are arbitrarily large line segments I *such that* f *is measurable on the straight line supporting* I *and*

$$\left|\int_I f\right| \ge C\sqrt{|I|}.$$

Proof of Corollary 1. Let $R > 0$ be large and let $G \subseteq T$ consist of all $t \in T$ such that

$$(t+Q)\cap[0,R]^2 \ne \emptyset.$$

Since Q is bounded the set G is finite. By the boundedness of Q we have that

$$\operatorname{diam} G = \sqrt{2}R + O(1), \tag{1}$$

as $R \to \infty$. For the same reason we have that

$$|G| = R^2 + O(R). \tag{2}$$

The lower order terms in (1) and (2) are due to the boundary.

We may now apply Theorem 1 to the set G and the coloring function

$$F(x) = \sum_{g\in G} z_g \chi_Q(x-g).$$

For sufficiently large R we obtain a straight line S such that F is measurable on S and

$$\begin{aligned}\left|\int_S F\right| &\ge C(\operatorname{diam} G)^{-1/2}|G|^{1/2}\\ &\ge C(R+O(1))^{-1/2}(R^2+O(R))^{1/2}\\ &\ge C\sqrt{R}.\end{aligned}$$

We may partition

$$S \cap \operatorname{supp} F = I \cup E,$$

where I is the line segment $S \cap [0,R]^2$ and the one-dimensional measure of the set E is at most a constant. It follows that

$$\begin{aligned}\left|\int_I f\right| &= \left|\int_I F\right|\\ &\ge \left|\int_S F\right| - O(1)\\ &\ge C\sqrt{R}\\ &\ge C\sqrt{|I|},\end{aligned}$$

as we had to show. □

1.4. **Does the checkerboard pattern matter?** It is a natural question whether the checkerboard pattern of the coloring function really matters for the discrepancy lower bounds. Could it be the case that for *any* measurable coloring function

$$f : [0,R]^2 \to \{-1,+1\}$$

there is always a straight line S, on which the restriction of f is measurable and

$$\left|\int_S f\right| \ge C\sqrt{R},$$

as is the case when the function f is a checkerboard coloring of 1×1 squares?

That this is not so can be proved using a (randomized) construction of [7]. In [7] it was proved that for every N there is a ± 1 coloring of the usual subdivision in unit squares of $[0,N]^2$ such that for any straight line segment I we have

$$\left|\int_I f\right| \le C\sqrt{N\log N}.$$

Take such a coloring of the square $[0, N]^2$. The discrepancy of any segment is at most $C \cdot \sqrt{N \log N}$. Now scale the coloring down to the square

$$\left[0, \sqrt{\frac{N}{\log N}}\right]^2,$$

so that it becomes a coloring of that square with a square checkerboard of side-length $\frac{1}{\sqrt{N \log N}}$. The discrepancy scales proportionally so any line segment in this new coloring now has discrepancy at most C.

2. Proof of the Main Theorem

Lemma 1. *Suppose $L > 0$ is a constant, $\Lambda \subseteq \mathbb{R}^d$ is a lattice and $g \in L^1(\mathbb{R}^d)$ is a nonnegative function, uniformly continuous on $\mathbb{R}^d$, such that*

$$\sum_{t \in \Lambda} g(x - t) = L, \tag{3}$$

for all $x \in \mathbb{R}^d$. Then for each $\epsilon > 0$ and for each $x \in \mathbb{R}^d$ there is $r = r(x, \epsilon) > 0$ and $R = R(x, \epsilon) > 0$ such that

$$\sum_{t \in \Lambda, |t| > R} g(y - t) < \epsilon, \qquad \textit{whenever } |y - x| < r. \tag{4}$$

Proof. For each $x \in \mathbb{R}^d$ it follows from (3) that there is $R = R(x, \epsilon) > 0$ such that

$$\sum_{t \in \Lambda, |t| > R} g(x - t) < \epsilon/2.$$

Let $N = \{t \in \Lambda : |t| \le R\}$. It follows that

$$\sum_{t \in N} g(x - t) > L - \epsilon/2. \tag{5}$$

This is a finite sum. Since g is uniformly continuous we can select $r = r(x, \epsilon) > 0$ such that

$$|x - y| < r \Rightarrow |g(x) - g(y)| < \frac{\epsilon}{2|N|}.$$

Combining this with (5) we obtain that whenever $|x - y| < r$ we have

$$\sum_{t \in N} g(y - t) > L - \epsilon,$$

and, because of (3), we have that (4) holds. $\square$

Corollary 2. *With the assumptions of Lemma 1, if $B \subseteq \mathbb{R}^d$ is a bounded set then for each $\epsilon > 0$ there is $R > 0$ such that*

$$\sum_{t \in \Lambda, |t| > R} g(x - t) < \epsilon, \qquad \textit{whenever } x \in B. \tag{6}$$

Proof. Let $\epsilon > 0$. The closure $\overline{B}$ of B is compact and is covered by the union of open balls

$$\bigcup_{x\in\overline{B}} B_{r(x,\epsilon)}(x),$$

where the radius $r(x,\epsilon)$ is given by Lemma 1. Therefore there exist $x_1,\ldots,x_n \in \overline{B}$ such that

$$\overline{B} \subseteq \bigcup_{j=1}^{n} B_{r(x_j,\epsilon)}(x_j).$$

Let $R = \max\{R(x_1,\epsilon), R(x_2,\epsilon),\ldots,R(x_n,\epsilon)\}$, where the numbers $R(x_j,\epsilon)$ are those given by Lemma 1, and the conclusion follows. ☐

Proof of Theorem 1. The definition of the Fourier Transform that we use is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x)e^{-2\pi i\xi\cdot x}\,dx$$

for $f \in L^1(\mathbb{R}^d)$ and $\xi \in \mathbb{R}^d$.

For a straight line L through the origin let us denote by $\pi_L f$ (the projection of f onto L) the function of $t \in \mathbb{R}$ given by

$$\pi_L f(t) = \int_{\mathbb{R}} f(tu + su^\perp)\,ds,$$

where u is a unit vector along L and $u^\perp$ is a unit vector orthogonal to u. By Fubini's theorem for any straight line L through the origin $\pi_L f(t)$ is well defined for almost all (Lebesgue) values of $t \in \mathbb{R}$.

It is well known and easy to see that the Fourier Transform of $\pi_L f$ is equal to the restriction of the (two-dimensional) Fourier Transform of f, $\widehat{f}$, on L:

$$\widehat{\pi_L f}(\xi) = \widehat{f}(\xi u),\quad \xi \in \mathbb{R}.$$

Write

$$M = \mathrm{esssup}_{\,L,t}|\pi_L f(t)|$$

where the essential supremum is taken over all lines L through the origin and real numbers t.

For the Fourier Transform of $f = \chi_Q * \sum_{g\in G} z_g\delta_g$ we have

$$\widehat{f}(\xi) = \widehat{\chi_Q}(\xi)\phi(\xi), \tag{7}$$

where the trigonometric polynomial

$$\phi(\xi) = \sum_{g\in G} z_g e^{-2\pi i g\xi}$$

is a T^*-periodic function. Here $T^* = \{x \in \mathbb{R}^2 :\ \langle x,t\rangle \in \mathbb{Z},\ \forall t \in T\}$ is the dual lattice of T, which also has area 1. It is also true that $T^{**} = T$.

If B is any measurable fundamental domain of T^* we have that $L^2(B)$ has the functions

$$e_t(x) = e^{-2\pi i t \cdot x}, \quad t \in T,$$

as an orthonormal basis. This is merely a restatement, after a linear transformation, of the fact that the exponentials with integer frequencies form an orthonormal basis of $L^2([0,1]^2)$. Parseval's identity then gives

$$\int_B |\phi(\xi)|^2 = \sum_{g \in G} |z_g|^2. \tag{8}$$

We are going to show that

$$M > C(\operatorname{diam} G)^{-1/2} \left(\sum_{g \in G} |z_g|^2 \right)^{1/2},$$

for some constant $C > 0$ that may depend only on the lattice T and the chosen fundamental domain Q.

Write $D = \operatorname{diam} G$. It follows that $\operatorname{diam} \operatorname{supp} f \le D + C$, for some constant C that depends on the fundamental domain Q. Since Q has area 1 it follows that D is bounded below by a constant and, therefore, we have that

$$\operatorname{diam} \operatorname{supp} f \le CD.$$

As we also have

$$\operatorname{diam} \operatorname{supp} \pi_L f \le CD$$

and $|\pi_L f(t)| \le M$ for almost all $t \in \mathbb{R}$ we obtain from Parseval's equality that

$$\int_{\mathbb{R}} \left| \widehat{f}(tu) \right|^2 dt = \int_{\mathbb{R}} |\pi_L f(t)|^2 \, dt \le CM^2 D. \tag{9}$$

It also follows from Parseval's equality that

$$\sum_{g \in G} |z_g|^2 = \int_{\mathbb{R}^2} |f|^2 = \int_{\mathbb{R}^2} \left| \widehat{f} \right|^2. \tag{10}$$

We now make the observation that since Q tiles the plane with the lattice T it follows that Q has the lattice T^* as a *spectrum* (see e.g. [6]). In other words, the family of exponentials

$$e_{t^*}(x) = e^{2\pi i t^* \cdot x}, \quad t^* \in T^*, \tag{11}$$

forms an orthonormal basis for $L^2(Q)$. This implies the tiling condition

$$\sum_{t^* \in T^*} g(\xi - t^*) = 1, \quad \text{for all } \xi \in \mathbb{R}^2, \tag{12}$$

where

$$g(\xi) = |\widehat{\chi_Q}(\xi)|^2 \text{ and } \int g = 1.$$

One can easily prove (12) by observing that the inner product, in $L^2(Q)$, of the function $e_\xi(t) = e^{2\pi i \xi\cdot x}$ and the function $e_{t^*}(x) = e^{2\pi i t^*\cdot x}$ is equal to

$$\widehat{\chi_Q}(\xi - t^*)$$

and then rewriting Parseval's identity for the function $e_\xi(t)$ with respect to the complete orthonormal system (11). The function $g(x)$ is uniformly continuous in $\mathbb{R}^2$ (since $\widehat{\chi_Q}$ is both uniformly continuous and bounded), therefore Lemma 1 and Corollary 2 are applicable to the function g and the lattice $\Lambda = T^*$.

Taking B to be a bounded fundamental domain of T^* and $\epsilon = 1/2$ we have therefore from Corollary 2 that there exists a finite $R > 0$ such that for $\xi \in B$ we have

$$\sum_{t^*\in T^*, |t^*|>R} g(\xi - t^*) < \frac{1}{2}. \tag{13}$$

Let now $R' > 0$ be such that

$$\{|\xi| > R'\} \subseteq \bigcup_{t^*\in T^*, |t^*|>R} (t^* + B). \tag{14}$$

This exists because B is a bounded set. We may assume that R' depends on T alone, by taking the fundamental domain B of T^* that minimizes R'.

We have

$$\begin{aligned}
\int_{|\xi|>R'} \left|\widehat{f}(\xi)\right|^2 &= \int_{|\xi|>R'} g(\xi)\left|\phi(\xi)\right|^2 \quad \text{(from (7))}\\
&\le \sum_{t^*\in T^*, |t^*|>R} \int_{t^*+B} g(\xi)\left|\phi(\xi)\right|^2 \quad \text{(from (14))}\\
&= \sum_{t^*\in T^*, |t^*|>R} \int_B g(\xi + t^*)\left|\phi(\xi)\right|^2 \quad \text{(since } \phi(\xi) \text{ is } T^*\text{-periodic)}\\
&= \int_B \left(\sum_{t^*\in T^*, |t^*|>R} g(\xi + t^*)\right)\left|\phi(\xi)\right|^2\\
&\le \frac{1}{2}\int_B \left|\phi(\xi)\right|^2 \quad \text{(from (13))}\\
&= \frac{1}{2}\sum_{g\in G} \left|z_g\right|^2 \quad \text{(from (8))}.
\end{aligned}$$

Writing S^1 for the unit circle in $\mathbb{R}^2$ it follows from (10) and the inequality just proved that

$$\begin{aligned}
\frac{1}{2}\sum_{g\in G} \left|z_g\right|^2 &\le \int_{|\xi|\le R'} \left|\widehat{f}(\xi)\right|^2\\
&= \int_{0\le t\le R'} \int_{u\in S^1} t\left|\widehat{f}(tu)\right|^2 du\, dt \quad \text{(in polar coordinates)}
\end{aligned}$$

$$\leq R' \int_{u\in S^1} \int_{\mathbb{R}} \left|\widehat{f}(tu)\right|^2 dt\, du$$
$$\leq R' 2\pi C M^2 D \quad \text{(from (9))}.$$

Solving for M in the inequality above, and remembering that R' depends only on Q and T, gives

$$M \geq C \frac{1}{\sqrt{D}} \left(\sum_{g\in G} \left|z_g\right|^2\right)^{1/2},$$

for C depending on Q and T only. □

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M.K.: Department of Mathematics and Aplied Mathematics, University of Crete, Voutes Campus., GR-700 13, Heraklion, Greece

E-mail address: kolount@gmail.com