On the structure of multiple translational tilings by polygonal regions

MIHAIL N. KOLOUNTZAKIS[1]
Department of Mathematics,
University of Crete,
Knossos Ave., 714 09 Iraklio,
Greece.

E-mail: kolount@math.uch.gr

February 1999

Abstract

We consider polygons with the following "pairing property": for each edge of the polygon there is precisely one other edge parallel to it. We study the problem of when such a polygon K tiles the plane multiply when translated at the locations Λ, where Λ is a multiset in the plane. The pairing property of K makes this question particularly amenable to Fourier Analysis. After establishing a necessary and sufficient condition for K to tile with a given lattice Λ (which was first found by Bolle for the case of convex polygons–notice that all convex polygons that tile, necessarily have the pairing property and, therefore, our theorems apply to them) we move on to prove that a large class of such polygons tiles only quasi-periodically, which for us means that Λ must be a finite union of translated 2-dimensional lattices in the plane. For the particular case of convex polygons we show that all convex polygons which are not parallelograms tile necessarily quasi-periodically, if at all.

§0. Introduction

In this paper we study multiple tilings of the plane by translates of a polygonal region of a certain type, the polygons with the pairing property of Definition 2 below.

Definition 1 (Tiling)
Let K be a measurable subset of $\mathbb{R}^2$ of finite measure and let $\Lambda \in \mathbb{R}^2$ be a discrete multiset (i.e., its underlying set is discrete and each point has finite multiplicity). We say that $K + \Lambda$ is a (translational, multiple) tiling of $\mathbb{R}^2$, if

$$\sum_{\lambda\in\Lambda} \mathbf{1}_K(x-\lambda) = w,$$

for almost all (Lebesgue) $x \in \mathbb{R}^2$, where the *weight* or *level* w is a positive integer and $\mathbf{1}_K$ is the indicator function of K.

Definition 2 (Polygons with the Pairing Property)
A polygon K has the pairing property if for each edge e there is precisely one other edge of K parallel to e

Remarks.
1. Note that all symmetric convex polygons have the pairing property and it is not hard to see that all convex polygons that tile by translation are necessarily symmetric.
2. The polygonal regions we deal with are not assumed to be connected.

[1]Partially supportedby the U.S. National Science Foundation, under grant DMS 97-05775.

Using Fourier Analysis we study the following two problems: (a) characterize the polygons that tile multiply with a lattice, and (b) determine which polygons tile necessarily in a "quasi-periodic" manner, if they tile at all. We restrict our attention to polygons with the pairing property.

Definition 3 (Quasi-periodic multisets)
A multiset $\Lambda \subseteq \mathbb{R}^d$ is called quasi-periodic if it is the union of finitely many d-dimensional lattices (see Definition 6) in $\mathbb{R}^d$.

In §1 we describe the general approach to translational tiling using the Fourier Transform of the indicator function of the tile and in particular its zero-set. This zero-set for polygons with the pairing property is calculated explicitly.

In §2 we give a necessary and sufficient condition (Theorem 2) for a polygon K with the pairing property to tile multiply with a lattice Λ. This has been proved previously by Bolle for the more special case of convex polygons (although his method might apply for the case of pairing polygons as well) who used a combinatorial method. Our approach is based on the calculation of §1.

In §3 we find a very large class of polygons with the pairing property that tile only in a quasi-periodic manner. In particular we show that every convex polygon that is not a parallelogram can tile (multiply) only in a quasi-periodic way.

Notation.
1. The Fourier Transform of a function $f \in L^1(\mathbb{R}^d)$ is normalized as follows:

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} e^{-2\pi i \langle \xi, x \rangle} f(x)\, dx.$$

It is extended to tempered distributions by duality.
2. The action of a tempered distribution α on a function ϕ of Schwarz class is denoted by $\alpha(\phi)$. The Fourier Transform $\widehat{\alpha}$ of α is defined by

$$\widehat{\alpha}(\phi) = \alpha(\widehat{\phi}).$$

A tempered distribution α is supported on a closed set K if for each smooth function ϕ with $\operatorname{supp}\phi \subset K^c$ we have $\alpha(\phi) = 0$. The intersection of all such closed sets K is called the support of α and denoted by $\operatorname{supp}\alpha$.

§1. The Fourier Analytic approach.

1.1 General

It is easy to see that if a polygon K with the pairing property tiles multiply then for each (relevant) direction the two edges parallel to it necessarily have the same length. For this, suppose that u is a direction and that e_1 and e_2 are the two edges parallel to it. Let then μ_u be the measure which is equal to arc-length on e_1 and negative arc-length on e_2. Suppose also that $K + \Lambda$ is a multiple tiling of $\mathbb{R}^2$. It follows then that

$$\sum_{\lambda \in \Lambda} \mu_u(x - \lambda)$$

is the zero measure in $\mathbb{R}^2$. This is so because each copy of edge e_1 in the tiling has to be countered be some copies of edge e_2. Hence the total mass of μ_u is 0 and e_1 and e_2 have the same length. We can then write (here e_1 and e_2 are viewed as point-sets in $\mathbb{R}^2$ and τ as a vector)

$$e_2 = e_1 + \tau,$$

for some $\tau \in \mathbb{R}^2$.

By the previous discussion, a polygon K with the pairing property tiles multiply with a multiset Λ if and only if for each pair e and $e+\tau$ of parallel edges of K

$$\sum_{\lambda \in \Lambda} \mu_e(x-\lambda) = 0, \tag{1}$$

where μ_e is the measure in $\mathbb{R}^2$ that is arc-length on e and negative arc-length on $e+\tau$. Write

$$\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda,$$

where δ_a is a unit point mass at a. Thus δ_Λ is locally a measure but is globally unbounded when Λ is infinite. However, whenever $K+\Lambda$ is a multiple tiling, it is obvious that Λ cannot have more than cR^2 points in any disc of radius R, $R>1$, (c depends on K and the weight of the tiling). This implies that δ_Λ is a tempered distribution and we can take its Fourier Transform, denoted by $\widehat{\delta_\Lambda}$. Condition (1) then becomes

$$\widehat{\mu_e} \cdot \widehat{\delta_\Lambda} = 0. \tag{2}$$

When Λ is a lattice $\Lambda = A\mathbb{Z}^2$, where A is a 2×2 invertible matrix, its <u>dual lattice</u> Λ^* is defined by

$$\Lambda^* = \left\{x \in \mathbb{R}^2 : \langle x, \lambda \rangle \in \mathbb{Z}, \forall \lambda \in \Lambda\right\},$$

and we have $\Lambda^* = A^{-\top}\mathbb{Z}^2$. The Poisson Summation Formula then takes the form

$$\widehat{\delta_\Lambda} = \det \Lambda \cdot \delta_{\Lambda^*}. \tag{3}$$

Since $\widehat{\mu_e}$ is a continuous function we have in this case, and whenever $\widehat{\delta_\Lambda}$ is locally a measure, that condition (2) is equivalent to

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq Z(\widehat{\mu_e}), \tag{4}$$

where for every continuous function f we write $Z(f)$ for the set where it vanishes. When Λ is a lattice (2) is equivalent to

$$\widehat{\mu_e}(x) = 0, \quad \forall x \in \Lambda^*.$$

So, to check if a given polygon K with the pairing property tiles multiply $\mathbb{R}^2$ with the lattice Λ, one has to check that $\widehat{\mu_e}$ vanishes on Λ^* for every edge e of K.

1.2 The shape of the zero-set

Here we study the zero-set of the Fourier Transform of the measure μ_e of §1.1 and determine its structure.

We first calculate the Fourier Transform of μ_e in the particular case when e is parallel to the x-axis, for simplicity. Let $\mu \in M(\mathbb{R}^2)$ be the measure defined by duality by

$$\mu(\phi) = \int_{-1/2}^{1/2} \phi(x, 0)\ dx, \quad \forall \phi \in C(\mathbb{R}^2).$$

That is, μ is arc-length on the line segment joining the points $(-1/2, 0)$ and $(1/2, 0)$. Calculation gives

$$\widehat{\mu}(\xi, \eta) = \frac{\sin \pi\xi}{\pi\xi}.$$

Notice that $\widehat{\mu}(\xi, \eta) = 0$ is equivalent to $\xi \in \mathbb{Z} \setminus \{0\}$.

If μ_L is the arc-length measure on the line segment joining $(-L/2, 0)$ and $(L/2, 0)$ we have

$$\widehat{\mu_L}(\xi, \eta) = \frac{\sin \pi L\xi}{\pi\xi}$$

and

$$Z(\widehat{\mu_L}) = \left\{(\xi, \eta) : \ \xi \in L^{-1}\mathbb{Z} \setminus \{0\}\right\}.$$

Write $\tau = (a, b)$ and let $\mu_{L,\tau}$ be the measure which is arc-length on the segment joining $(-L/2, 0)$ and $(L/2, 0)$ translated by $\tau/2$ and negative arc-length on the same segment translated by $-\tau/2$. That is, we have

$$\mu_{L,\tau} = \mu_L * (\delta_{\tau/2} - \delta_{-\tau/2}),$$

and, taking Fourier Transforms, we get

$$\widehat{\mu_{L,\tau}}(\xi, \eta) = -2\frac{\sin \pi L\xi}{\pi\xi} \sin \pi(a\xi + b\eta).$$

Define $u = \frac{\tau}{|\tau|^2}$ and $v = (1/L, 0)$. It follows that

$$Z(\widehat{\mu_{L,\tau}}) = (\mathbb{Z}u + \mathbb{R}u^\perp) \cup (\mathbb{Z} \setminus \{0\}v + \mathbb{R}v^\perp).$$

This a set of straight lines of direction $u^\perp$ spaced by $|u|$ and containing 0 plus a similar set of lines of direction $v^\perp$, spaced by v and containing zero. However in the latter set of parallel lines the straight line through 0 has been removed. We state this as a theorem for later use, formulated in a coordinate-free way.

Definition 4 (Geometric inverse of a vector)
The geometric inverse of a non-zero vector $u \in \mathbb{R}^d$ is the vector

$$u^* = \frac{u}{|u|^2}.$$

Theorem 1 *Let e and $e + \tau$ be two parallel line segments (translated by τ, of magnitude and direction described by e, symmetric with respect to 0). Let also $\mu_{e,\tau}$ be the measure which charges e with its arc-length and $e + \tau$ with negative its arc-length. Then*

$$Z(\widehat{\mu_{e,\tau}}) = (\mathbb{Z}\tau^* + \mathbb{R}\tau^{*\perp}) \cup (\mathbb{Z} \setminus \{0\}e^* + \mathbb{R}e^{*\perp}). \tag{5}$$

§2. When does a polygon tile with a certain lattice?

The following theorem has been proved by Bolle [**Bo94**] who used combinatorial methods.

Theorem (Bolle)
A convex polygon K, which is centrally symmetric about 0, tiles multiply with the lattice Λ (for some weight $w \in \mathbb{N}$) if and only if for each edge e of K the following two conditions are satisfied.

- (i) In the relative interior of e there is a point of $\frac{1}{2}\Lambda$, and
- (ii) If the midpoint of e is not in $\frac{1}{2}\Lambda$ then the vector e is in Λ.

Remark. Notice that Bolle's theorem implies that all convex polygons with vertices in Λ tile multiply with Λ at some level.

We prove the following which is easily seen to be a generalization of Bolle's Theorem to polygons with the pairing property.

Theorem 2 *If the polygon K has the pairing property and Λ is a lattice in $\mathbb{R}^2$ then $K+\Lambda$ is a multiple tiling of $\mathbb{R}^2$ if and only if for each pair of edges e and $e+\tau$ of K*

- *(i) $\tau \in \Lambda$, or*
- *(ii) $e \in \Lambda$ and $\tau + \theta e \in \Lambda$, for some $0 < \theta < 1$.*

Proof of Theorem 2. Once again we simplify matters and take the edge e to be parallel to the x-axis and follow the notation of §1.1.

For an arbitrary non-zero vector $w \in \mathbb{R}^2$ define the group

$$G(w) = \mathbb{Z}w + \mathbb{R}w^\perp,$$

which is a set of straight lines in $\mathbb{R}^2$ of direction $w^\perp$ spaced regularly at distance $|w|$. It follows that

$$Z(\widehat{\mu_{L,\tau}}) \subseteq G(u) \cup G(v).$$

From Theorem 1 it follows that $\Lambda^* \subseteq Z(\widehat{\mu_{L,\tau}})$ which implies that $\Lambda^* \subset G(u)$ or $\Lambda^* \subset G(v)$.

This is a consequence of the following.

Observation 1 If G, H, K are groups and $G \subseteq H \cup K$ then $G \subseteq H$ or $G \subseteq K$.

For, if $a \in G \setminus K$ and $b \in G \setminus H$, then $a \cdot b \in H$, say, which implies $b \in H$, a contradiction.

So we have the two alternatives

1. $\Lambda^* \subset G(u)$,
2. $\Lambda^* \subset G(v)$.

However, since not all of $G(v)$ is in $Z(\widehat{\mu_{L,\tau}})$, if alternative 2 holds and alternative 1 does not, it follows that

$$\Lambda^* \subseteq \mathrm{span}_{\mathbb{Z}}\{v, w\}, \tag{6}$$

where w is the smallest (in length) multiple of $v^\perp$ which is in $G(u)$, i.e.,

$$w = (0, 1/b).$$

We have that (6) is equivalent to

$$\Lambda \supseteq (\mathrm{span}_{\mathbb{Z}}\{v, w\})^* = \mathbb{Z}(L, 0) + \mathbb{Z}(0, b),$$

which is in turn equivalent to

$$(L, 0) \in \Lambda \quad \text{and} \quad (0, b) \in \Lambda.$$

Notice also that

$$\Lambda^* \subseteq G(u) \Longleftrightarrow \Lambda \supseteq G(u)^* \Longleftrightarrow \Lambda \ni \frac{u}{|u|^2} = \tau.$$

We have therefore proved the following lemma.

Lemma 1 *If Λ is a lattice, $u = \frac{(a,b)}{a^2+b^2}$ and $v = (L,0)$, then*

$$\Lambda^* \subset (\mathbb{Z}u + \mathbb{R}u^\perp) \cup (\mathbb{Z}\setminus\{0\}v + \mathbb{R}v^\perp)$$

if and only if

1. $(a,b) \in \Lambda$, *or*
2. $(L,0) \in \Lambda$ *and* $(0,b) \in \Lambda$.

Allowing for a general linear transformation, let $\tau, e \in \mathbb{R}^2$, and let $\mu_{e,\tau}$ be the measure that "charges" with its arc-length the line segment e translated so that its midpoint is at $\tau/2$ and charges with negative its arc-length the line segment e with its midpoint at $-\tau/2$. We have proved the following:

$$\Lambda^* \subset Z(\widehat{\mu_{e,\tau}}) \Longleftrightarrow \left\{ \begin{array}{l} \tau \in \Lambda, \quad \text{or} \\ e \in \Lambda \quad \text{and} \quad \tau + \theta e \in \Lambda, \quad \text{for some } 0 < \theta < 1. \end{array} \right. \tag{7}$$

This completes the proof of Theorem 2.
■

§3. Polygons that tile only quasi-periodically

3.1 Meyer's theorem

We now deal with the following question: which polygons with the pairing property admit only quasi-periodic multiple tilings. The main tool here, as was in [**KL96**], is the idempotent theorem of P.J. Cohen for general locally compact abelian groups, in the form of the following theorem of Y. Meyer [**M70**].

Definition 5 (The coset ring)
The coset ring of an abelian group G is the smallest collection of subsets of G which is closed under finite unions, finite intersections and complements (that is, the smallest ring of subsets of G) and which contains all cosets of G

Remark. When the group is equipped with a topology one usually only demands that the open cosets of G are in the coset ring, but we take all cosets in our definition.

Theorem (Meyer)
Let $\Lambda \subseteq \mathbb{R}^d$ be a discrete set and δ_Λ be the Radon measure

$$\delta_\Lambda = \sum_{\lambda\in\Lambda} c_\lambda \delta_\lambda, \quad c_\lambda \in S,$$

where $S \subseteq \mathbb{C}\setminus\{0\}$ is a finite set. Suppose that δ_Λ is tempered, and that $\widehat{\delta_\Lambda}$ is a Radon measure on $\mathbb{R}^d$ which satisfies

$$\left|\widehat{\delta_\Lambda}\right|([-R,R]^d) \le CR^d, \text{ as } R \to \infty, \tag{8}$$

where $C > 0$ is a constant. Then, for each $s \in S$, the set

$$\Lambda_s = \{\lambda \in \Lambda : \ c_\lambda = s\}$$

is in the coset ring of $\mathbb{R}^d$.

A proof of Meyer's theorem for $d = 1$ can be found in [**KL96**]. The proof works verbatim for all d.

3.2 Discrete elements of the coset ring

In this section we determine the structure of the discrete elements of the coset ring of $\mathbb{R}^d$.

In dimension $d = 1$ we have the following characterization of the discrete elements of the coset ring of $\mathbb{R}$, due to Rosenthal [**R66**].

Theorem (Rosenthal)
The elements of the coset ring of $\mathbb{R}$ which are discrete in the usual topology of $\mathbb{R}$ are precisely the sets of the form

$$F \bigtriangleup \bigcup_{j=1}^{J} (\alpha_j \mathbb{Z} + \beta_j) \ , \qquad (9)$$

where $F \subseteq \mathbb{R}$ is finite, $\alpha_j > 0$ and $\beta_j \in \mathbb{R}$ ($\bigtriangleup$ denotes symmetric difference).

Rosenthal's proof does not extend to dimension $d \geq 2$. Since we need to know what kind of sets the elements of the coset ring of $\mathbb{R}^2$ are, we prove the following general theorem.

Theorem 3 *Let G be a topological abelian group and let $\mathcal{R}$ be the least ring of sets which contains the discrete cosets of G. Then $\mathcal{R}$ contains all discrete elements of the coset ring of G.*

In other words, a discrete element of the coset ring can always be written as a finite union of sets of the type

$$A_1 \cap \cdots \cap A_m \cap B_1^c \cap \cdots \cap B_n^c, \qquad (10)$$

where the A_i and B_i are discrete cosets of G. And, observing that the intersection of any two cosets is a coset, we may rewrite (10) as

$$A \cap B_1^c \cap \cdots \cap B_n^c, \qquad (11)$$

where A and all B_i are discrete cosets.

We need the following lemma.

Lemma 2 *Suppose that A is a non-discrete topological abelian group, $F \subset A$ is discrete and $B_1, \ldots, B_m$ are cosets in A disjoint from F. Then*

$$A = F \cup B_1 \cup \cdots \cup B_n \qquad (12)$$

implies that $F = \emptyset$. This remains true if A is a coset in a larger group.

Proof of Lemma 2. Write $B_i = x_i + G_i$ and let k be the number of different subgroups G_i appearing in (12). We do induction on k. Notice that the group G_1 may be assumed to be non-discrete, by the non-discreteness of A.

When $k = 1$ the theorem is true as then F is a union of cosets of G_1 and cannot be discrete unless it is empty. (Here is where the disjointness of F from the B_i is used.)

Assume the theorem true for $k \leq n$ and suppose that precisely $n + 1$ groups appear in (12) and that $F \neq \emptyset$. Assume that the G_1-cosets in (12) are

$$x_1 + G_1, \ldots, x_r + G_1,$$

and let $y \in F$. We then have

$$y + G_1 \subseteq F \cup (X_2 + G_2) \cup \cdots \cup (X_{n+1} + G_{n+1}),$$

with all sets X_i, $i = 2, \ldots, n+1$, being finite. Hence

$$\begin{aligned} G_1 &\subseteq (-y + F) \cup (-y + X_2 + G_2) \cup \cdots \cup (-y + X_{n+1} + G_{n+1}) \\ &= F' \cup (X'_2 + G_2) \cup \cdots \cup (X'_{n+1} + G_{n+1}), \end{aligned}$$

with $F' = -y + F$, $X'_i = -y + X_i$.

Furthermore, one may take $X'_i \subset G_1$, $i = 2, \ldots, n+1$ (possibly empty), to get

$$G_1 \subseteq (F' \cap G_1) \cup (X'_2 + G_2 \cap G_1) \cup \cdots \cup (X'_{n+1} + G_{n+1} \cap G_1).$$

Since $y \in F$ we have that $F' \cap G_1 \ni 0$ (hence it is non-empty) and

$$(F' \cap G_1) \cap (X'_i + G_i \cap G_1) = \emptyset, \quad i = 2, \ldots, n+1.$$

By the induction hypothesis we get a contradiction.
■

Proof of Theorem 3. By Lemma 2, if A is non-discrete then $A \cap B_1^c \cap \cdots \cap B_n^c$ is either non-discrete or empty. Hence a finite union of such sets can only be discrete if all participating A's are discrete. Rewrite then

$$A \cap B_1^c \cap \cdots \cap B_n^c = A \cap (B_1 \cap A)^c \cap \cdots \cap (B_n \cap A)^c$$

so as to have the arbitrary discrete element of the coset ring made up with finitely many operations from discrete cosets.
■

Definition 6 (Dimension, lattices)
The dimension of a set $A \subseteq \mathbb{R}^d$ is the dimension of the smallest translated subspace of $\mathbb{R}^d$ that contains A. A lattice is a discrete subgroup of $\mathbb{R}^d$.

Remark. It is well known that all k-dimensional lattices in $\mathbb{R}^d$ are of the form $A\mathbb{Z}^k$, where A is a $d \times k$ real matrix of rank k.

Theorem 4 *Let $C = A \cap B_1^c \cap \cdots \cap B_n^c$, with A, B_i being discrete cosets of $\mathbb{R}^d$. Then C may be written as a finite (possibly empty) union of sets of the type*

$$K \cap L_1^c \cap \cdots \cap L_m^c, \quad L_i \subseteq K \subseteq A, \quad m \geq 0, \tag{13}$$

where the K, L_i are discrete cosets and, when C is not empty,

$$\dim L_i < \dim K = \dim A = \dim C.$$

Observation 2 If A and B are discrete cosets in $\mathbb{R}^d$ with $\dim A = \dim B = \dim A \cap B$ then $A \cap B^c$ is a finite (possibly empty) union of disjoint cosets of $A \cap B$ and, therefore, $\dim A \cap B^c = \dim A$, except when $A \cap B^c = \emptyset$. Hence A and B can each be written as a finite disjoint union of translates of $A \cap B$.

Proof of Theorem 4: Notice that

$$C = A \cap (B_1 \cap A)^c \cap \cdots \cap (B_n \cap A)^c.$$

Let

$$\alpha = \dim A = \dim B_1 \cap A = \cdots = \dim B_r \cap A$$

and $\dim B_i \cap A < \alpha$ for $i > r \geq 0$. Let

$$C' = A \cap (B_1 \cap A)^c \cap \cdots \cap (B_r \cap A)^c.$$

By induction on $r \geq 0$ we prove that C' is a finite union of sets of type (13). For $r = 0$ this is obvious. If it is true for $r - 1$ then C' is a finite union of sets of type

$$K \cap L_1^c \cap \cdots \cap L_m^c \cap (B_r \cap A)^c,$$

with $\alpha = \dim K > \dim L_i$, $i = 1, \ldots, m$. Each of these sets falls into one of two categories:
Category 1: $\dim K \cap (B_r \cap A) = \alpha$.
Then, by Observation 2 above, $K \cap (B_r \cap A)^c$ is a finite union of cosets $K_1, \ldots, K_s$ of dimension α and hence C' is a finite union of $K_i \cap L_1^c \cap \cdots \cap L_m^c$.
Category 2: $\dim K \cap (B_r \cap A) < \alpha$.
Then

$$K \cap L_1^c \cap \cdots \cap L_m^c \cap (B_r \cap A)^c$$

is already of the desired form.
■

From Theorems 3 and 4 it follows for $d = 2$ that every discrete element S of the coset ring of $\mathbb{R}^2$ may be written as

$$S = \left(\bigcup_{j=1}^{J} A_j \setminus (B_1^{(j)} \cup \cdots \cup B_{n_j}^{(j)}) \right) \cup \bigcup_{l=1}^{L} L_l \triangle F, \tag{14}$$

where $A_1, \ldots, A_J$ are 2-dimensional translated lattices, L_l and $B_i^{(j)}$ are 1-dimensional translated lattices and F is a finite set ($J, L \geq 0$). And, repeatedly using Observation 2, the lattices A_j may be assumed to be have pairwise intersections of dimension at most 1.

3.3 Purely discrete Fourier Transform

Definition 7 (Uniform density)
A multiset $\Lambda \subseteq \mathbb{R}^d$ has asymptotic density ρ if

$$\lim_{R \to \infty} \frac{|\Lambda \cap B_R(x)|}{|B_R(x)|} \to \rho$$

uniformly in $x \in \mathbb{R}^d$.
We say that Λ has (uniformly) bounded density if the fraction above is bounded by a constant ρ uniformly for $x \in \mathbb{R}$ and $R > 1$. We say then that Λ has density uniformly bounded by ρ.

Assume that $\Lambda \subset \mathbb{R}^2$ is a discrete multiset of bounded density which satisfies the assumptions of Meyer's Theorem (if we write c_λ for the multiplicity of $\lambda \in \Lambda$). Then, if Λ_k is the subset of Λ of multiplicity k, Λ_k is a discrete element of the coset ring and is of the form (14).

Assume now in addition that $\widehat{\delta_\Lambda}$ has discrete support. We shall prove that all sets F, L_l and $B_i^{(j)}$ are empty in (14) and so

$$\Lambda = \bigcup_{j=1}^{J} A_j,$$

where the A_i are translated 2-dimensional lattices in $\mathbb{R}^2$.

One can easily show that whenever $\Omega \subseteq \mathbb{R}^d$ of finite measure tiles with Λ at level w then Λ has density $w/|\Omega|$.

Theorem 5 *Suppose that $\Lambda \in \mathbb{R}^d$ is a multiset with density ρ, $\delta_\Lambda = \sum_{\lambda\in\Lambda}\delta_\lambda$, and that $\widehat{\delta_\Lambda}$ is a measure in a neighborhood of 0. Then $\widehat{\delta_\Lambda}(\{0\}) = \rho$.*

Proof of Theorem 5. Take $\phi \in C^\infty$ of compact support with $\phi(0) = 1$. We have

$$\begin{aligned}
\widehat{\delta_\Lambda}(\{0\}) &= \lim_{t\to\infty} \widehat{\delta_\Lambda}(\phi(tx)) \\
&= \lim_{t\to\infty} \delta_\Lambda(t^{-d}\widehat{\phi}(\xi/t)) \\
&= \lim_{t\to\infty} t^{-d}\sum_{\lambda\in\Lambda}\widehat{\phi}(\lambda/t) \\
&= \lim_{t\to\infty} \sum_{n\in\mathbb{Z}^d}\sum_{\lambda\in Q_n} t^{-d}\widehat{\phi}(\lambda/t)
\end{aligned}$$

where, for fixed and large $T > 0$,

$$Q_n = [0,T)^d + Tn, \qquad n \in \mathbb{Z}^d.$$

Since Λ has density ρ it follows that for each $\epsilon > 0$ we can choose T large enough so that for all n

$$|\Lambda \cap Q_n| = \rho|Q_n|(1+\delta_n),$$

with $|\delta_n| \le \epsilon$. For each n and $\lambda \in Q_n$ we have

$$\widehat{\phi}(\lambda/t) = \widehat{\phi}(Tn/t) + r_\lambda$$

with $|r_\lambda| \le CTt^{-1}\left\|\nabla\widehat{\phi}\right\|_{L^\infty(t^{-1}Q_n)}$. Hence

$$\begin{aligned}
\widehat{\delta_\Lambda}(\{0\}) &= \lim_{t\to\infty}\sum_{n\in\mathbb{Z}^d} t^{-d}\sum_{\lambda\in Q_n}(\widehat{\phi}(Tn/t) + r_\lambda) \\
&= \lim_{t\to\infty}\sum_{n\in\mathbb{Z}^d} t^{-d}\rho|Q_n|(1+\delta_n)\widehat{\phi}(Tn/t) + \\
&\quad \lim_{t\to\infty}\sum_{n\in\mathbb{Z}^d} t^{-d}\sum_{\lambda\in Q_n} r_\lambda \\
&= \lim_{t\to\infty} S_1 + \lim_{t\to\infty} S_2.
\end{aligned}$$

We have

$$\left|S_1 - \sum_n t^{-d}\rho|Q_n|\widehat{\phi}(Tn/t)\right| \le \epsilon\sum_n t^{-d}\rho|Q_n|\left|\widehat{\phi}(Tn/t)\right| \tag{15}$$

The first sum in (15) is a Riemann sum for $\rho\int_{\mathbb{R}^d}\widehat{\phi} = \rho$ and the second is a Riemann sum for $\rho\int_{\mathbb{R}^d}\left|\widehat{\phi}\right| < \infty$.

For S_2 we have

$$\begin{aligned}
|S_2| &\le C\sum_{n\in\mathbb{Z}^d} t^{-d}\rho|Q_n|(1+\delta_n)Tt^{-1}\left\|\nabla\widehat{\phi}\right\|_{L^\infty(t^{-1}Q_n)} \\
&\le \rho CTt^{-1}\sum_{n\in\mathbb{Z}^d} t^{-d}|Q_n|\left\|\nabla\widehat{\phi}\right\|_{L^\infty(t^{-1}Q_n)}.
\end{aligned}$$

The sum above is a Riemann sum for $\int_{\mathbb{R}^d} \left|\nabla\widehat{\phi}\right|$, which is finite, hence $\lim_{t\to\infty} S_2 = 0$.

Since ϵ is arbitrary the proof is complete.

■

Remark: The same proof as that of Theorem 5 shows that, if

$$\mu = \sum_{\lambda\in\Lambda} c_\lambda \delta_\lambda,$$

with $|c_\lambda| \le C$, Λ is of density 0 and the tempered distribution $\widehat{\mu}$ is locally a measure in the neighborhood of some point $a \in \mathbb{R}^2$, then we have $\widehat{\mu}(\{a\}) = 0$.

Theorem 6 *Suppose that $\Lambda \subset \mathbb{R}^2$ is a uniformly discrete multiset and that*

$$\widehat{\delta_\Lambda} = \left(\sum_{\lambda\in\Lambda} \delta_\lambda\right)^\wedge$$

is locally a measure with

$$\left|\widehat{\delta_\Lambda}\right|(B_R(0)) \le CR^2,$$

for some positive constant C. Assume also that $\widehat{\delta_\Lambda}$ has discrete support. Then Λ is a finite union of translated lattices.

Proof of Theorem 6. Define the sets (not multisets)

$$\Lambda_k = \{\lambda \in \Lambda : \ \lambda \text{ has multiplicity } k\}.$$

By Meyer's Theorem (applied for the base set of the multiset Λ with the coefficients c_λ equal to the corresponding multiplicities) each of the Λ_k is in the coset ring of $\mathbb{R}^2$ and, being discrete, is of the type (14).

We may thus write

$$\Lambda_k = A \bigtriangleup B, \tag{16}$$

with $A = \bigcup_{j=1}^J A_j$, where the 2-dimensional translated lattices A_j have pairwise intersections of dimension at most 1, and $\operatorname{dens} B = 0$.

Hence

$$\delta_{\Lambda_k} = \sum_{j=1}^J \delta_{A_j} + \mu,$$

where $\mu = \sum_{f\in F} c_f \delta_f$, $\operatorname{dens} F = 0$ and $|c_f| \le C(J)$. The set F consists of B and all points contained in at least two of the A_j.

Combining for all k, and reusing the symbols A_j, μ and F we get

$$\delta_\Lambda = \sum_{j=1}^J \delta_{A_j} + \mu.$$

But $\widehat{\delta_\Lambda}$ and $\sum_{j=1}^J \widehat{\delta_{A_j}}$ are both (by the assumption and the Poisson Summation Formula) discrete measures, and so is therefore $\widehat{\mu}$. However $\operatorname{dens} F = 0$ and the boundedness of the coefficients c_f implies that $\widehat{\mu}$ has no point masses (see the Remark after the proof of Theorem 5), which means that $\widehat{\mu} = 0$ and so is μ. Hence $\delta_\Lambda = \sum_{j=1}^J \delta_{A_j}$, or

$$\Lambda = \bigcup_{j=1}^J A_j, \quad \text{as multisets.}$$

■

Finally, we show that discrete support for $\widehat{\delta_\Lambda}$ implies that $\widehat{\delta_\Lambda}$ is locally a measure.

Theorem 7 *Suppose that the multiset $\Lambda \subset \mathbb{R}^d$ has density uniformly bounded by ρ and that, for some point $a \in \mathbb{R}^d$ and $R > 0$,*

$$\operatorname{supp} \widehat{\delta_\Lambda} \cap B_R(a) = \{a\}.$$

Then, in $B_R(a)$, we have $\widehat{\delta_\Lambda} = w\delta_a$, for some $w \in \mathbb{C}$ with $|w| \le \rho$.

Proof of Theorem 7. It is well known that the only tempered distributions supported at a point a are finite linear combinations of the derivatives of δ_a. So we may assume that, for $\phi \in C^\infty(B_R(a))$,

$$\widehat{\delta_\Lambda}(\phi) = \sum_\alpha c_\alpha (D^\alpha \delta_a)(\phi) = \sum_\alpha (-1)^{|\alpha|} c_\alpha D^\alpha \phi(a), \tag{17}$$

where the sum extends over all values of the multiindex $\alpha = (\alpha_1, \ldots, \alpha_d)$ with $|\alpha| = \alpha_1 + \cdots + \alpha_d \le m$ (the finite degree) and $D^\alpha = \partial_1^{\alpha_1} \cdots \partial_d^{\alpha_d}$ as usual.

We want to show that $m = 0$. Assume the contrary and let α_0 be a multiindex that appears in (17) with a non-zero coefficient and has $|\alpha_0| = m$. Pick a smooth function ϕ supported in a neighborhood of 0 which is such that for each multiindex α with $|\alpha| \le m$ we have $D^\alpha \phi(0) = 0$ if $\alpha \ne \alpha_0$ and $D^{\alpha_0}\phi(0) = 1$. (To construct such a ϕ, multiply the polynomial $1/\alpha_0! x^{\alpha_0}$ with a smooth function supported in a neighborhood of 0, which is identically equal to 1 in a neighborhood of 0.)

For $t \to \infty$ let $\phi_t(x) = \phi(t(x-a))$. Equation (17) then gives that

$$\widehat{\delta_\Lambda}(\phi_t) = t^m (-1)^m c_{\alpha_0}. \tag{18}$$

On the other hand, using

$$(\phi(t(x-a)))^\wedge(\xi) = e^{-2\pi i \langle a, \xi/t\rangle} t^{-d} \widehat{\phi}(\xi/t)$$

we get

$$\widehat{\delta_\Lambda}(\phi_t) = \sum_{\lambda \in \Lambda} e^{-2\pi i \langle a, \lambda/t\rangle} t^{-d} \widehat{\phi}(\lambda/t). \tag{19}$$

Notice that (19) is a bounded quantity as $t \to \infty$ by a proof similar to that of Theorem 5, while (18) increases like t^m, a contradiction.

Hence $\widehat{\delta_\Lambda} = w\delta_a$ in a neighborhood of a. The proof of Theorem 5 again gives that $|w| \le \rho$.

■

Using Theorem 7 we may drop from Theorem 6 the assumption that $\widehat{\delta_\Lambda}$ has to be locallly a measure, as this is now implied by the discrete support which we assume for $\widehat{\delta_\Lambda}$. Summing up we have the following.

Theorem 8 *Suppose that the multiset Λ has uniformly bounded density, that $S = \operatorname{supp} \widehat{\delta_\Lambda}$ is discrete, and that*

$$|S \cap B_R(0)| \le CR^d,$$

for some positive constant C. Then Λ is a finite union of translated d-dimensional lattices.

3.4 Application to tilings by polygons

In this section we apply Theorem 8 and the characterization of the zero-sets of the functions $\widehat{\mu_{e,\tau}}$ (Theorem 1) in order to give very general sufficient conditions for a polygon K to admit only quasi-periodic tilings, if it tiles at all.

Theorem 9 *Let the polygon K have the pairing property and tile multiply the plane with the multiset Λ. Denote the edges of K by (we follow the notation of §1.1)*

$$e_1, e_1 + \tau_1, e_2, e_2 + \tau_2, \ldots, e_n, e_n + \tau_n.$$

Suppose also that

$$\{\widetilde{e_1}, \widetilde{\tau_1}\} \cap \cdots \cap \{\widetilde{e_n}, \widetilde{\tau_n}\} = \emptyset, \tag{20}$$

where with $\widetilde{v}$ we denote the orientation of vector v. Then Λ is a finite union of translated 2-dimensional lattices.

Proof of Theorem 9. By Theorem 1 and the tiling assumption we get

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq Z(\widehat{\mu_{e_1,\tau_1}}) \cap \cdots \cap Z(\widehat{\mu_{e_n,\tau_n}}).$$

By Theorem 1 in the intersection above each of the sets is contained in a collection of lines in the direction $\widetilde{e_i}$ union a collection of lines in the direction $\widetilde{\tau_i}$. Because of assumption (20) these sets have a discrete intersection as two lines of different orientations intersect at a point. Furthermore, because of the regular spacing of these pairs of sets of lines, it follows that the resulting intersection has at most CR^2 points in a large disc of radius R. Theorem 8 now implies that Λ is a finite union of translated 2-dimensional lattices.
■

The condition (20) is particularly easy to check for convex polygons.

Theorem 10 *Suppose that K is a symmetric convex polygon which is not a parallelogram. Then K admits only quasi-periodic multiple tilings.*

Proof of Theorem 10. Suppose that (20) fails and that the intersection in (20) contains a vector which is, say, parallel to the x-axis. It follows that each pair of edges $e_i, e_i + \tau_i$ of edges of K either (a) has both edges parallel to the x-axis, or (b) has the line joining the two midpoints parallel to the x-axis. As this latter line goes through the origin it is clear that (b) can only happen for one pair of edges and, since (a) cannot happen for two consecutive pairs of edges, (a) can hold at most once as well. This means that K is a parallelogram.
■

Remarks.
1. It is clear that parallelograms admit tilings which are not quasi-periodic. Take for example the regular tiling by a square and move each vertical column of squares arbitrarily up or down.
2. Some very interseting classes of polygons are left out of reach of Theorem 9. An important class consists of all polygons whose edges are parallel to either the x- or the y-axis.

§4. Bibliography

[**Bo94**] U. Bolle, On multiple tiles in E^2, in *Intuitive Geometry*, K. Böröczky and G. Fejes Tóth, editors, Colloq. Math. Soc. J. Bolyai 63, 1994.

[**K68**] Y. Katznelson, *An introduction to harmonic analysis*, Wiley (1968) and Dover (1976).

[**KL96**] M.N. Kolountzakis and J.C. Lagarias, Structure of tilings of the line by a function, Duke Math. J. **82** (1996), 3, 653-678.

[**M70**] Y. Meyer, Nombres de Pisot, nombres de Salem et analyse harmonique, Lect. Notes Math. **117**(1970), Springer-Verlag.

[**R66**] H.P. Rosenthal, Projections onto translation-invariant subspaces of $L^P(G)$, Memoirs Amer. Math. Soc. **63** (1966).