A class of non-convex polytopes that admit no orthonormal basis of exponentials

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Abstract

A conjecture of Fuglede states that a bounded measurable set $\Omega \subset \mathbb{R}^d$, of measure 1, can tile $\mathbb{R}^d$ by translations if and only if the Hilbert space $L^2(\Omega)$ has an orthonormal basis consisting of exponentials $e_\lambda(x) = \exp 2\pi i \langle \lambda, x \rangle$. If Ω has the latter property it is called *spectral*. Let Ω be a polytope in $\mathbb{R}^d$ with the following property: there is a direction $\xi \in S^{d-1}$ such that, of all the polytope faces perpendicular to ξ, the total area of the faces pointing in the positive ξ direction is more than the total area of the faces pointing in the negative ξ direction. It is almost obvious that such a polytope Ω cannot tile space by translation. We prove in this paper that such a domain is also not spectral, which agrees with Fuglede's conjecture. As a corollary, we obtain a new proof of the fact that a convex body that is spectral is necessarily symmetric, in the case where the body is a polytope.

Let Ω be a measurable subset of $\mathbb{R}^d$, which we take for convenience to be of measure 1. Let also Λ be a discrete subset of $\mathbb{R}^d$. We write

$$\begin{aligned} e_\lambda(x) &= \exp 2\pi i \langle \lambda, x \rangle, \quad (\lambda, x \in \mathbb{R}^d), \\ E_\Lambda &= \{ e_\lambda : \ \lambda \in \Lambda \} \subset L^2(\Omega). \end{aligned}$$

The inner product and norm on $L^2(\Omega)$ are

$$\langle f, g \rangle_\Omega = \int_\Omega f \overline{g}, \quad \text{and} \quad \|f\|_\Omega^2 = \int_\Omega |f|^2.$$

Definition 1 The pair (Ω, Λ) is called a *spectral pair* if E_Λ is an orthonormal basis for $L^2(\Omega)$. A set Ω will be called *spectral* if there is $\Lambda \subset \mathbb{R}^d$ such that (Ω, Λ) is a spectral pair. The set Λ is then called a *spectrum* of Ω.

Example: If $Q_d = (-1/2, 1/2)^d$ is the cube of unit volume in $\mathbb{R}^d$ then $(Q_d, \mathbb{Z}^d)$ is a spectral pair (d-dimensional Fourier series).

We write $B_R(x) = \{ y \in \mathbb{R}^d : \ |x - y| < R \}$.

Definition 2 (Density)
The discrete set $\Lambda \subset \mathbb{R}^d$ has *density* ρ, and we write $\rho = \operatorname{dens} \Lambda$, if we have

$$\rho = \lim_{R \to \infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|},$$

uniformly for all $x \in \mathbb{R}^d$.

We define translational tiling for complex-valued functions below.

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Definition 3 Let $f : \mathbb{R}^d \to \mathbb{C}$ be measurable and $\Lambda \subset \mathbb{R}^d$ be a discrete set. We say that f *tiles with* Λ *at level* $w \in \mathbb{C}$, and sometimes write "$f + \Lambda = w\mathbb{R}^d$", if

$$\sum_{\lambda \in \Lambda} f(x - \lambda) = w, \quad \text{for almost every (Lebesgue) } x \in \mathbb{R}^d, \tag{1}$$

with the sum above converging absolutely a.e. If $\Omega \subset \mathbb{R}^d$ is measurable we say that $\Omega + \Lambda$ *is a tiling* when $\mathbf{1}_\Omega + \Lambda = w\mathbb{R}^d$, for some w. If w is not mentioned it is understood to be equal to 1.

Remark 1
If $f \in L^1(\mathbb{R}^d)$, $f \geq 0$, and $f + \Lambda = w\mathbb{R}^d$, then the set Λ has density

$$\operatorname{dens} \Lambda = \frac{w}{\int f}.$$

The following conjecture is still unresolved in all dimensions and in both directions.

Conjecture: (Fuglede [**F74**]) If $\Omega \subset \mathbb{R}^d$ is bounded and has Lebesgue measure 1 then $L^2(\Omega)$ has an orthonormal basis of exponentials if and only if there exists $\Lambda \subset \mathbb{R}^d$ such that $\Omega + \Lambda = \mathbb{R}^d$ is a tiling.

Fuglede's conjecture has been confirmed in several cases.

1. Fuglede [**F74**] shows that if Ω tiles with Λ being a lattice then it is spectral with the dual lattice Λ^* being a spectrum. Conversely, if Ω has a lattice Λ as a spectrum then it tiles by the dual lattice Λ^*.

2. If Ω is a convex non-symmetric domain (bounded, open set) then, as the first author of the present paper has proved [**K00**], it cannot be spectral. It has long been known that convex domains which tile by translation must be symmetric.

3. When Ω is a smooth convex domain it is clear that it admits no translational tilings. Iosevich, Katz and Tao [**IKT**] have shown that it is also not spectral.

4. There has also been significant progress in dimension 1 (the conjecture is still open there as well) by Laba [**La**, **Lb**]. For example, the conjecture has been proved in dimension 1 if the domain Ω is the union of two intervals.

In this paper we describe a wide class of, generally non-convex, polytopes for which Fuglede's conjecture holds.

Theorem 1 *Suppose Ω is a polytope in $\mathbb{R}^d$ with the following property: there is a direction $\xi \in S^{d-1}$ such that*

$$\sum_i \sigma^*(\Omega_i) \neq 0.$$

The finite sum is extended over all faces Ω_i of Ω which are orthogonal to ξ and $\sigma^*(\Omega_i) = \pm\sigma(\Omega_i)$, where $\sigma(\Omega_i)$ is the surface measure of Ω_i and the $\pm$ sign depends upon whether the outward unit normal vector to Ω_i is in the same or opposite direction with ξ.

Then Ω is not spectral.

Such polytopes cannot tile space by translation for the following, intuitively clear, reason. In any conceivable such tiling the set of positive-looking faces perpendicular to ξ must be countered by an equal area of negatively-looking ξ-faces, which is impossible because there is more (say) area of the former than the latter.

The following corollary is a special case of the result in [**K00**], which says that all spectral convex domains are symmetric.

Corollary 1 *If Ω is a spectral convex polytope then it is necessarily symmetric.*

Proof. If Ω is spectral, then, from Theorem 1 the area measure of Ω is symmetric (see [**S**] for the definition of the area measure).

This implies that Ω is itself symmetric, as the area measure determines a convex body up to translation [**S**, Th. 4.3.1], and therefore Ω and $-\Omega$ which have the same surface measure are translates of each other.

□

It has been observed in recent work on this problem (see e.g. [**K00**]) that a domain (of volume 1) is spectral with spectrum Λ if and only if $|\widehat{\chi_\Omega}|^2 + \Lambda$ is a tiling of Euclidean space at level 1. By Remark 1 this implies that Λ has density 1.

By the orthogonality of e_λ and e_μ for any two different λ and μ in Λ, it follows that

$$\widehat{\chi_\Omega}(\lambda - \mu) = 0. \tag{2}$$

It is only this property, and the fact that any spectrum of Ω must have density 1, that are used in the proof.

Proof of Theorem 1.

The quantities P, Q, N, ℓ and K, which are introduced in the proof below, will depend only on the domain Ω. (The letter K will denote several different constants.)

Suppose that Λ is a spectrum of Ω. Define the Fourier transform of χ_Ω as

$$\widehat{\chi_\Omega}(\eta) = \int_\Omega e^{-2\pi i \langle x, \eta \rangle}\, dx.$$

By an easy application of the divergence theorem we get

$$\widehat{\chi_\Omega}(\eta) = -\frac{1}{i|\eta|} \int_{\partial\Omega} e^{-2\pi i \langle x, \eta \rangle} \left\langle \frac{\eta}{|\eta|}, \nu(x) \right\rangle d\sigma(x), \quad \eta \neq 0,$$

where $\nu(x) = (\nu_1(x), \ldots, \nu_d(x))$ is the outward unit normal vector to $\partial\Omega$ at $x \in \partial\Omega$ and $d\sigma$ is the surface measure on $\partial\Omega$.

From the last formula we easily see that for some $K \geq 1$

$$|\nabla \widehat{\chi_\Omega}(\eta)| \leq \frac{K}{|\eta|}, \quad |\eta| \geq 1. \tag{3}$$

Without loss of generality we assume that $\xi = (0, \ldots, 0, 1)$. Hence

$$\widehat{\chi_\Omega}(t\xi) = -\frac{1}{it} \int_{\partial\Omega} e^{-2\pi i t x_d} \nu_d(x)\, d\sigma(x).$$

Now it is easy to see that each face of the polytope other than any of the Ω_is contributes $O(t^{-2})$ to $\widehat{\chi_\Omega}(t\xi)$ as $t \to \infty$. Therefore

$$\left|\widehat{\chi_\Omega}(t\xi) + \frac{1}{it}\sum_i e^{-2\pi i\lambda_i t}\sigma^*(\Omega_i)\right| \le \frac{K}{t^2}, \quad t \ge 1, \tag{4}$$

where λ_i is the value of x_d for $x = (x_1, \ldots, x_d) \in \Omega_i$.

Now define

$$f(t) = \sum_i \sigma^*(\Omega_i) e^{-2\pi i\lambda_i t}, \quad t \in \mathbb{R}.$$

f is a finite trigonometric sum and has the following properties:

(i) f is an almost-periodic function.

(ii) $f(0) \ne 0$ by assumption. Without loss of generality assume $f(0) = 1$.

(iii) $|f'(t)| \le K$, for every $t \in \mathbb{R}$.

By (i), for every $\epsilon > 0$ there exists an $\ell > 0$ such that every interval of $\mathbb{R}$ of length ℓ contains a translation number τ of f belonging to ϵ:

$$\sup_t |f(t+\tau) - f(t)| \le \epsilon \tag{5}$$

(see [**B32**]).

Fix $\epsilon > 0$ to be determined later ($\epsilon = 1/6$ will do) and the corresponding ℓ. Fix the partition of $\mathbb{R}$ in consecutive intervals of length ℓ, one of them being $[0, \ell]$. Divide each of these ℓ-intervals into N consecutive equal intervals of length ℓ/N, where

$$N > \frac{6K\ell\sqrt{d-1}}{\epsilon}.$$

In each ℓ-interval there is at least one $\frac{\ell}{N}$-interval containing a number τ satisfying (5). For example, in $[0, \ell]$ we may take $\tau = 0$ and the corresponding $\frac{\ell}{N}$-interval to be $[0, \ell/N]$.

Define the set L to be the union of all these $\frac{\ell}{N}$-intervals in $\mathbb{R}$. Then $L\xi$ is a copy of L on the x_d-axis. Construct M by translating copies of the cube $[0, \ell/N]^d$ along the x_d-axis so that they have their x_d-edges on the $\frac{\ell}{N}$-intervals of $L\xi$.

The point now is that there can be no two λs of Λ in the same translate of M, at distance $D > \frac{2K}{\epsilon}$ from each other. Suppose, on the contrary, that

$$\lambda_1, \lambda_2 \in \Lambda, \quad |\lambda_1 - \lambda_2| \ge D, \quad \lambda_1, \lambda_2 \in M + \eta.$$

Then $\lambda_1 = t_1\xi + \eta + \eta_1$, $\lambda_2 = t_2\xi + \eta + \eta_2$, for some $t_1, t_2 \in L$, $\eta_1, \eta_2 \in \mathbb{R}^d$ with

$$|\eta_1|, |\eta_2| < \frac{\ell}{N}\sqrt{d-1} < \frac{\epsilon}{6K}.$$

Hence, $\lambda_1 - \lambda_2 = (t_1 - t_2)\xi + \eta_1 - \eta_2$ and an application of the mean value theorem together with (2) and (3) gives

$$|\widehat{\chi_\Omega}((t_1 - t_2)\xi)| \le \frac{3K}{|t_1 - t_2|}|\eta_1 - \eta_2|.$$

From (4) we get

$$|f(t_1 - t_2)| \le 3K|\eta_1 - \eta_2| + \frac{K}{|t_1 - t_2|} < 2\epsilon.$$

Now, since $t_1, t_2 \in L$, there exist τ_1, τ_2 satisfying (5) so that

$$|\tau_1 - t_1|, |\tau_2 - t_2| < \frac{\ell}{N}$$

and hence (by (iii))

$$|f(\tau_1 - \tau_2) - f(\tau_1 - t_2)|, |f(\tau_1 - t_2) - f(t_1 - t_2)| < K\frac{\ell}{N} < \epsilon.$$

Therefore

$$\begin{aligned} 2\epsilon &> |f(t_1 - t_2)| \\ &\ge |f(0)| - |f(0) - f(-\tau_2)| - |f(-\tau_2) - f(\tau_1 - \tau_2)| \\ &\quad - |f(\tau_1 - \tau_2) - f(\tau_1 - t_2)| - |f(\tau_1 - t_2) - f(t_1 - t_2)| \\ &\ge 1 - \epsilon - \epsilon - \epsilon - \epsilon. \end{aligned}$$

It suffices to take $\epsilon = 1/6$ for a contradiction.

Therefore, as the distance between any two λs is bounded below by the modulus of the zero of $\widehat{\chi_\Omega}$ that is nearest to the origin, there exists a natural number P so that every translate of M contains at most P elements of Λ and, hence, there exists a natural number Q (we may take $Q = 2NP$) so that every translate of

$$\mathbb{R}\xi + [0, \frac{\ell}{N}]^d$$

contains at most Q elements of Λ.

It follows that Λ cannot have positive density, a contradiction as any spectrum of Ω (which has volume 1) must have density equal to 1.

□

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