MULTIPLE LATTICE TILES AND RIESZ BASES OF EXPONENTIALS

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ABSTRACT. Suppose $\Omega \subseteq \mathbb{R}^d$ is a bounded and measurable set and $\Lambda \subseteq \mathbb{R}^d$ is a lattice. Suppose also that $\Omega$ tiles multiply, at level $k$, when translated at the locations $\Lambda$. This means that the $\Lambda$-translates of $\Omega$ cover almost every point of $\mathbb{R}^d$ exactly $k$ times. We show here that there is a set of exponentials $\exp(2\pi it \cdot x)$, $t \in T$, where $T$ is some countable subset of $\mathbb{R}^d$, which forms a Riesz basis of $L^2(\Omega)$. This result was recently proved by Grepstad and Lev under the extra assumption that $\Omega$ has boundary of measure 0, using methods from the theory of quasicrystals. Our approach is rather more elementary and is based almost entirely on linear algebra. The set of frequencies $T$ turns out to be a finite union of shifted copies of the dual lattice $\Lambda^*$. It can be chosen knowing only $\Lambda$ and $k$ and is the same for all $\Omega$ that tile multiply with $\Lambda$.

Keywords: Riesz bases of exponentials; Tiling

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Notation: We write $e(t) = e^{2\pi i t}$. If $E$ is a set then $\chi_E$ is its indicator function. If $A$ is a non-singular $d \times d$ matrix and $\Lambda = AZ^d$ is a lattice in $\mathbb{R}^d$ then $\Lambda^* = A^{-\top}Z^d$ denotes the dual lattice.

1. Introduction

1.1. Riesz bases. In this paper we deal with the question of existence of a Riesz (unconditional) basis of exponentials

$$e_t(x) := e(t \cdot x) = e^{2\pi i t \cdot x}, \quad t \in L,$$

for the space $L^2(\Omega)$, where $\Omega \subseteq \mathbb{R}^d$ is a domain of finite Lebesgue measure and $L \subseteq \mathbb{R}^d$ is a countable set of frequencies. By Riesz basis we mean that every $f \in L^2(\Omega)$ can be written uniquely in the form

$$f(x) = \sum_{t \in L} a_t \cdot e(t \cdot x)$$

with the coefficients $a_t \in \mathbb{C}$ satisfying

$$C_1 \|f\|_2^2 \leq \sum_{t \in L} |a_t|^2 \leq C_2 \|f\|_2^2,$$

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for some positive and finite constants $C_1, C_2$.

1.2. Orthogonal bases. One very special example of a Riesz basis occurs when the exponentials $e(t \cdot x), t \in L$, can be chosen to be orthogonal and complete for $L^2(\Omega)$. One can then choose $a_t = ||\Omega||^{-1/2} \langle f, e_t \rangle$ and $C_1 = C_2 = 1/||\Omega||$ for (2) to hold as an equality. For instance, if $\Omega = (0, 1)^d$ is the unit cube in $\mathbb{R}^d$ then one can take $L = \mathbb{Z}^d$ and obtain such an orthogonal basis of exponentials. This case, where an orthogonal basis of exponentials exists, is a very rigid situation though and many “reasonable” domains do not have such a basis (a ball is one example [4, 10], or any other smooth convex body or any non-symmetric convex body [7]).

The problem of which domains admit an orthogonal basis of exponentials has been studied intensively. The so called Fuglede or Spectral Set Conjecture [4] (claiming that for $\Omega$ to have such a basis it is necessary and sufficient that it can tile space by translations) was eventually proved to be false in dimension at least 3 [20, 12, 2, 3, 11], in both directions. Yet the conjecture may still be true in several important special cases such as convex bodies [8], and it generated many interesting results even after the disproof of its general validity (a rather dated account may be found in [10]).

It is expected that the existence of a Riesz basis for a domain $\Omega$ is a much more general, and perhaps even generic, phenomenon, although proofs of existence of a Riesz basis for specific domains are still rather rare, especially in higher dimension [13, 14, 16]. Also no domain is known not to have a Riesz basis of exponentials [13].

1.3. Lattice tiles. One general class of domains for which an orthogonal basis of exponentials is known to exist is the class of lattice tiles. A domain $\Omega \in \mathbb{R}^d$ is said to tile space when translated at the locations of the lattice $L$ (a discrete additive subgroup of $\mathbb{R}^d$ containing $d$ linearly independent vectors) if

$$\sum_{t \in L} \chi_\Omega(x - t) = 1, \quad \text{for almost all } x \in \mathbb{R}^d.$$

Intuitively this condition means that one can cover $\mathbb{R}^d$ with the $L$-translates of $\Omega$, with no overlaps, except for a set of measure zero (usually the translates of $\partial \Omega$, for “nice” domains $\Omega$).

It is not hard to see that when $\Omega$ has finite and non-zero measure then the set $L$ has density equal to $1/||\Omega||$. If $L$ is a lattice then we call $\Omega$ an almost fundamental domain of $L$ and $||\Omega|| = (\text{dens } L)^{-1}$. A fundamental domain of $L$ is any set which contains exactly one element of each coset mod $L$, for instance a fundamental parallelepiped. There are of course many others, as indicated in Figure 1.

![Figure 1: Shaded $\Omega$ is a fundamental domain of $\mathbb{R}^2/\mathbb{Z}^2$](image)

It is easy to see [4, 10] that every lattice tile by the lattice $L$ has an orthogonal basis of exponentials, namely those with frequencies $t \in L^*$, where $L^*$ is the dual lattice.
1.4. Multiple tiling by a lattice. We say that a domain tiles multiply when its translates cover space the same number of times, almost everywhere.

**Definition 1.1.** Let $\Omega \subseteq \mathbb{R}^d$ be measurable and $L \subseteq \mathbb{R}^d$ be a countable set. We say that $\Omega$ tiles $\mathbb{R}^d$ when translated by $L$ at level $k \in \mathbb{N}$ if

$$\sum_{t \in L} \chi_\Omega(x-t) = k,$$

for almost every $x \in \mathbb{R}^d$. If we do not specify $k$ then we mean $k = 1$.

Multiple tiles are a much wider class of domains than level-one tiles. For instance [1, 9], any centrally symmetric convex polygon in the plane whose vertices have integer coordinates tiles multiply by the lattice $\mathbb{Z}^2$ at some level $k \in \mathbb{N}$. In contrast, only parallelograms or symmetric hexagons can tile at level one.

Another difference is the fact that if two disjoint domains $\Omega_1$ and $\Omega_2$ both tile multiply when translated at the locations $L$ then so does their union. In the case of multiple lattice tiling this operation gives essentially the totality of multiple tiles starting from level-one tiles, according to the following easy Lemma.

**Lemma 1.** Suppose $\Omega \subseteq \mathbb{R}^d$ is a measurable set which tiles $\mathbb{R}^d$ at level $k$ when translated by the lattice $\Lambda \subseteq \mathbb{R}^d$. Then we can partition

$$\Omega = \Omega_1 \cup \cdots \cup \Omega_k \cup E,$$

where $E$ has measure 0 and the $\Omega_j$ are measurable, mutually disjoint and each $\Omega_j$ is an almost fundamental domain of the lattice $\Lambda$.

**Proof.** Let $D \subseteq \mathbb{R}^d$ be a measurable fundamental domain of $\Lambda$, for instance one of its fundamental parallelepipeds. For almost every $x \in D$ (call the exceptional set $E \subseteq D$) it follows from our tiling assumption that $\Omega \cap (x + \Lambda)$ contains exactly $k$ points, which we denote by

$$p_1(x) < p_2(x) < \cdots < p_k(x),$$

ordered according to the lexicographical ordering in $\mathbb{R}^d$. We also have that almost every point of $\Omega$ belongs to exactly one such list.

Let then $\Omega_j = \bigcup_{x \in D \setminus E} p_j(x)$, for $j = 1, 2, \ldots, k$. In other words, for (almost) each one of the classes mod $\Lambda$ we distribute its $k$ occurrences in $\Omega$ into the sets $\Omega_j$. It is easy to see that the $\Omega_j$ are disjoint and measurable and that they are almost fundamental domains of $\Lambda$. \qed

1.5. **Multiple lattice tiles have Riesz bases of exponentials.** It is not true that domains that tile multiply by a lattice have an orthogonal basis of exponentials. For instance, it is known [8] that the only convex polygons that have such a basis are parallelograms and symmetric hexagons, yet every symmetric convex polygon with integer vertices is a multiple tile, a much wider class.

It is however true that multiple tiles have a Riesz basis of exponentials. The main result of this paper is the following theorem.

**Theorem 1.** Suppose $\Omega \subseteq \mathbb{R}^d$ is bounded, measurable and tiles $\mathbb{R}^d$ multiply at level $k$ with the lattice $\Lambda$. Then there are vectors $a_1, \ldots, a_k \in \mathbb{R}^d$ such that the exponentials

$$e\left((a_j + \lambda^*) \cdot x\right), \ j = 1, 2, \ldots, k, \ \lambda^* \in \Lambda^*$$

form a Riesz basis for $L^2(\Omega)$.

The vectors $a_1, \ldots, a_k$ depend on $\Lambda$ and $k$ only, not on $\Omega$. 
Theorem 1 was proved by Grepstad and Lev [6] with the additional topological assumption that the boundary $\partial \Omega$ has Lebesgue measure 0.

In [6] the result is proved following the method of [18, 17] on quasicrystals. Our approach is more elementary and almost entirely based on linear algebra. The authors of [6] have pointed out to me that there are similarities of the method in this paper and the methods in [14, 15, 16]. The method essentially appears also in [19, §3.2].

As an interesting corollary of Theorem 1 let us mention, as is done in [6], that, according to the recent result of [5], if $\Omega$ is a centrally symmetric polytope in $\mathbb{R}^d$, whose codimension 1 faces are also centrally symmetric and whose vertices all have rational coordinates, then $L^2(\Omega)$ has a Riesz basis of exponentials.

**Open Problem 1.** Is Theorem 1 still true if $\Omega$ is of finite measure but unbounded?

### 2. Proof of the main result

The essence of the proof is contained in the following lemma.

**Lemma 2.** Suppose $\Omega \subseteq \mathbb{R}^d$ is bounded, measurable and tiles $\mathbb{R}^d$ multiply at level $k$ with the lattice $\Lambda$. Then there exist vectors $a_1, a_2, \ldots, a_k \in \mathbb{R}^d$ such that the following is true.

1. The $f_j$ are $\Lambda$-periodic,
2. The $f_j$ are in $L^2$ of any almost fundamental domain of $\Lambda$, and
3. We have the decomposition
   \[ f(x) = \sum_{j=1}^{k} e(a_j \cdot x) f_j(x), \quad \text{for a.e. } x \in \Omega. \]

Finally we have

\[ C_1 \|f\|_{L^2(\Omega)}^2 \leq \sum_{j=1}^{k} \|f_j\|_{L^2(\Omega)}^2 \leq C_2 \|f\|_{L^2(\Omega)}^2, \]

where $0 < C_1, C_2 < \infty$ do not depend on $f$.

**Proof.** Using Lemma 1 we can write $\Omega$ as the disjoint union

\[ \Omega = \Omega_1 \cup \cdots \cup \Omega_k, \]

where each $\Omega_k$ is a measurable almost fundamental domain of $\Lambda$. We can now define for $j = 1, 2, \ldots, k$ and for almost every $x \in \mathbb{R}^d$

\[ \omega_j(x) \text{ as the unique point in } \Omega_j \text{ s.t. } x - \omega_j(x) \in \Lambda, \text{ and} \]

\[ \lambda_j(x) = x - \omega_j(x). \]

(The maps $\omega_j$ are clearly measurable and measure-preserving when restricted to a fundamental domain of $\Lambda$.) Since the sought-after $f_j$ are to be $\Lambda$-periodic it is enough to define them on $\Omega_1$ and extend them to $\mathbb{R}^d$ by their $\Lambda$-periodicity. We may therefore rewrite our target decomposition (7) equivalently as follows.

\[ \text{For each } x \in \Omega_1 \text{ and } r = 1, 2, \ldots, k: f(\omega_r(x)) = \sum_{j=1}^{k} e(a_j \cdot (x - \lambda_r(x))) f_j(x). \]

We view (11) as a $k \times k$ linear system

\[ \tilde{M} \tilde{F} = F \]

whose right-hand side is the column vector

\[ F = (f(\omega_1(x)), f(\omega_2(x)), \ldots, f(\omega_k(x)))^T \]
and the unknowns form the column vector 
\[ \vec{F} = (f_1(x), f_2(x), \ldots, f_k(x))^\top. \]

We have a different linear system for each \( x \in \Omega_1 \) and its matrix is \( M = M(x) \in \mathbb{C}^{k \times k} \) with

\[ M_{r,j} = M_{r,j}(x) = e \left( a_j \cdot (x - \lambda_r(x)) \right), \quad r, j = 1, 2, \ldots, k. \tag{13} \]

Factoring we can write this matrix as

\[ M(x) = N(x) \text{ diag} (e (a_1 \cdot x), e (a_2 \cdot x), \ldots, e (a_k \cdot x)), \] with the matrix \( N = N(x) \) given by

\[ N_{r,j} = N_{r,j}(x) = e \left( -a_j \cdot \lambda_r(x) \right), \quad r, j = 1, 2, \ldots, k. \tag{14} \]

The key observation here is that when varying \( x \in \Omega_1 \) the number of different \( N(x) \) matrices that arise (the \( a_j \) are fixed) is finite and bounded by a quantity that depends on \( \Omega \) and \( \Lambda \) only. The reason for this is that the vectors \( \lambda_r(x) \) are among the \( \Lambda \) vectors in the bounded set \( \Omega - \Omega \), hence they take values in a finite set. (This is the only place where the boundedness of \( \Omega \) is used.)

Let us now see that the vectors \( a_1, \ldots, a_k \) can be chosen so that all the (finitely many) possible matrices \( N \) are invertible. We have

\[ \det N(x) = \sum_{\pi \in S_k} \text{sgn}(\pi) e \left( -\sum_{j=1}^k a_j \cdot \lambda_{\pi_j}(x) \right), \tag{15} \]

where \( S_k \) denotes the permutation group on \( \{1, 2, \ldots, k\} \). By the definition of the vectors \( \lambda_r(x) \) and the disjointness of the sets \( \Omega_r \) it follows that for each \( x \) no two \( \lambda_r(x) \) can be the same. View now the expression (15) as a function of the vector \( a = (a_1, \ldots, a_k) \in \mathbb{R}^{dk} \). Clearly it is a trigonometric polynomial and it is not identically zero as all the frequencies (for \( \pi \) in the symmetric group \( S_k \))

\[ \lambda_{\pi}(x) = (\lambda_{\pi_1}(x), \ldots, \lambda_{\pi_k}(x)) \in \mathbb{R}^{dk}, \tag{16} \]

are distinct precisely because all the \( \lambda_r(x) \) are distinct. Since the zero-set of any trigonometric polynomial (that is not identically zero) is a set of codimension at least 1 it follows that the vectors \( a_1, \ldots, a_k \) can be chosen so that all the \( N(x) \) matrices that arise are invertible.

Let now \( x \in \Omega_1 \) and consider the solution of the linear system (12) at \( x \) that now takes the form

\[ \tilde{F}(x) = \text{diag} (e (-a_1 \cdot x), e (-a_2 \cdot x), \ldots, e (-a_k \cdot x)) N(x)^{-1} F(x). \tag{17} \]

Since \( N(x) \) runs through a finite number of invertible matrices and the diagonal matrix in (17) is an isometry it follows that there are finite constants \( A_1, A_2 > 0 \), independent of \( f \), such that for any \( x \in \Omega_1 \) we have

\[ A_1 \| F(x) \|_{L^2}^2 \leq \| \tilde{F}(x) \|_{L^2}^2 \leq A_2 \| F(x) \|_{L^2}^2. \tag{18} \]

Integrating (18) over \( \Omega_1 \) we obtain

\[ A_1 \| f \|_{L^2(\Omega)}^2 \leq \sum_{j=1}^k \| f_j \|_{L^2(\Omega)}^2 \leq A_2 \| f \|_{L^2(\Omega)}^2. \tag{19} \]

This implies (8) with \( C_j = k \cdot A_j, j = 1, 2 \). To show the uniqueness of the decomposition (7) observe that any such decomposition must satisfy the linear system (17), whose non-singularity has been ensured by our choice of the \( a_j \). \( \square \)

We can now complete the proof of our main result.
Proof of Theorem 1. Let \( f \in L^2(\Omega) \). By Lemma 2 we can write \( f \) as in (7). Since the \( f_j \) are \( \Lambda \)-periodic and are in \( L^2 \) of any almost fundamental domain \( D \) of \( \Lambda \) it follows that we can expand each \( f_j \) in the frequencies of \( \Lambda^* \) (the dual lattice of \( \Lambda \))

\[
(20) \quad f_j(x) = \sum_{\lambda^* \in \Lambda^*} f_{j,\lambda^*} e(\lambda^* \cdot x), \quad j = 1, 2, \ldots, k,
\]

with

\[
(21) \quad \|f_j\|_{L^2(D)}^2 = \sum_{\lambda^* \in \Lambda^*} |f_{j,\lambda^*}|^2,
\]

since the exponentials \( e(\lambda^* \cdot x), \lambda^* \in \Lambda^* \), form an orthogonal basis of \( L^2(D) \) (we assume without loss of generality that \( |D| = 1 \).

The completeness of (6) follows from (7):

\[
(22) \quad f(x) = \sum_{j=1}^k \sum_{\lambda^* \in \Lambda^*} f_{j,\lambda^*} e\left((a_j + \lambda^*) \cdot x\right).
\]

The fact that (6) is a Riesz sequence follows from (8):

\[
\frac{k}{C_2} \sum_{j,\lambda^*} |f_{j,\lambda^*}|^2 \leq \left\| \sum_{\lambda^*} f_{j,\lambda^*} e\left((a_j + \lambda^*) \cdot x\right) \right\|^2_{L^2(\Omega)} \leq \frac{k}{C_1} \sum_{\lambda^*} |f_{j,\lambda^*}|^2.
\]

As is clear from the proof above, the \( k \)-tuples of vectors \( a_1, \ldots, a_k \) that appear in Theorem 1 are a generic choice: almost all \( k \)-tuples will do. The exceptional set in \( \mathbb{R}^{dk} \) is a set of lower dimension.

With a little more care one can see that one can choose the vectors \( a_1, \ldots, a_k \) to depend on \( \Lambda \) and \( k \) only and not on \( \Omega \). In the proof of Lemma 2 the \( a_j \) were chosen to ensure that the trigonometric polynomials (15) are all non-zero. Fix \( \Lambda \) and \( k \) and form the set of all polynomials of the form (15) which are not identically zero. This set of polynomials is countable and each such polynomial vanishes on a set of codimension at least 1 in \( \mathbb{R}^{dk} \). It follows that the union of their zero sets cannot possibly exhaust \( \mathbb{R}^{dk} \) and we only have to choose the \( a_j \) to avoid that union.

Thus there is a choice of \( a_j \) that works for all \( \Omega \) of the same lattice. This proof does not give uniform values for the constants \( C_1 \) and \( C_2 \) in (8) though. \( \square \)

References


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