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Lattice-tiling properties of integral self-affine functions

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Abstract

Let A be a $d \times d$ expanding integer matrix and $\rho : \mathbf{Z}^d \to \mathbf{C}$ be absolutely summable and satisfy $\sum_{x \in \mathbf{Z}^d} \rho(x) = |\det A|$. A function $f \in L^1(\mathbf{R}^d)$ is called an *integral self-affine function* for the pair (A, ρ) if it satisfies the functional equation $f(A^{-1}x) = \sum_{y \in \mathbf{Z}^d} \rho(y) f(x-y)$, a.e. (x). We prove that for such a function there is always a sublattice Λ of $\mathbf{Z}^d$ such that f tiles $\mathbf{R}^d$ with Λ with weight $w = \left|\mathbf{Z}^d : \Lambda\right|^{-1} \int_{\mathbf{R}^d} f$. That is $\sum_{\lambda \in \Lambda} f(x - \lambda) = w$, a.e. (x). The lattice $\Lambda \subseteq \mathbf{Z}^d$ is the smallest A-invariant sublattice of $\mathbf{Z}^d$ that contains the support of ρ. This generalizes results of Lagarias and Wang [**LaW**] and others, which were obtained for f and ρ which are indicator functions of compact sets. The proofs use Fourier Analysis.

§0. Integral self-affine functions

Let $A \in M_d(\mathbf{Z})$ be a $d \times d$ non-singular integer matrix which is also *expanding*, i.e., all its eigenvalues have modulus larger than 1. Let also $\rho : \mathbf{Z}^d \to \mathbf{C}$ be absolutely summable and satisfy

$$\sum_{x \in \mathbf{Z}^d} \rho(x) = |\det A|.$$

We generalize the terminology of Hutchinson [**Hu**] and Lagarias and Wang [**LaW**].

Definition 1 *A measurable function $f : \mathbf{R}^d \to \mathbf{C}$ is an* integral self-affine function *corresponding to the pair (A, ρ) if it has the property*

$$f(A^{-1}x) = \sum_{y \in \mathbf{Z}^d} \rho(y) f(x-y), \quad \textit{for a.e. } x \in \mathbf{R}^d. \qquad (1)$$

As an example, in dimension 1, the function $f(x) = \max\{1 - |x|, 0\}$ is an integral self-affine function with $A = [2]$ and

$$\rho(x) = \begin{cases} 1 & \text{if } x = 0 \\ 1/2 & \text{if } x = \pm 1 \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Lambda = \Lambda(A, \rho)$ be the smallest A-invariant sublattice of $\mathbf{Z}^d$ (i.e., $A\Lambda \subseteq \Lambda$) which contains

$$\mathcal{D} := \operatorname{supp} \rho = \left\{ y \in \mathbf{Z}^d : \ \rho(y) \neq 0 \right\}.$$

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Definition 2 *The complex-valued function* $f \in L^1(\mathbf{R}^d)$ tiles $\mathbf{R}^d$ *with the discrete* tile set $A \subset \mathbf{R}^d$ *and* weight $w \in \mathbf{C}$ *if*

$$\sum_{a \in A} f(x-a) = w, \quad a.e.\ (x),$$

with the series converging absolutely.

In our definition of tiling the tiles may "overlap". This means that we admit tilings with weights different from 1 (even for tiles f which are indicator functions of sets), as was done in [**KLa**].

The purpose of this note is to give a short proof of the following result about self-affine functions $f \in L^1(\mathbf{R}^d)$.

Theorem 1 *Let* $f \in L^1(\mathbf{R}^d)$ *be an integral self-affine function corresponding to the pair* (A, ρ). *Then* f tiles $\mathbf{R}^d$ *with* $\Lambda(A, \rho)$ *and weight*

$$w = \left|\mathbf{Z}^d : \Lambda\right|^{-1} \cdot \int_{\mathbf{R}^d} f(x)\ dx. \tag{2}$$

This generalizes a result of Lagarias and Wang [**LaW**], in which the result was proved under the additional assumptions that

(i) f is the indicator function of a compact set $T \subset \mathbf{R}^d$, and

(ii) ρ takes the values 0 and 1 only.

In this special case the set $\mathcal{D}$ is called a *digit set* and has $|\det A|$ elements. The set T is then called an *integral self-affine tile* for the pair $(A, \mathcal{D})$ and satisfies the set-theoretic equation

$$AT = \bigcup_{d \in \mathcal{D}} (T + d),$$

with the translates $T + d$, $d \in \mathcal{D}$, overlapping at most on a set of measure 0.

The reader should consult [**LaW**, **LaW2**] where the state of our knowledge about integral self-affine tiles, as well as their uses, is discussed. Let us only mention here that an indicator function of an integral self-affine tile T can be used [**GM**] as a scaling function of a mutliresolution analysis for orthonormal wavelets, provided that T tiles with the lattice $\mathbf{Z}^d$ (at weight 1).

The following is immediate from Theorem 1. It was also proved in [**LaW**] under the extra assumptions (i) and (ii) mentioned above.

Corollary 1 *Let* $f \in L^1(\mathbf{R}^d)$ *be an integral self-affine function for the pair* (A, ρ) *which is the indicator function of some measurable set* $T \subset \mathbf{R}^d$ *of finite measure. Then* $|T|$ *is an integer multiple of the index* $\left|\mathbf{Z}^d : \Lambda(A, \rho)\right|$.

Notice that we do not require the range of ρ to be $\{0, 1\}$.

Proof. The weight w of any tiling of f with Λ must obviously be an integer. Use (2).
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It is well known [**LaW**] that an integral self-affine tile T which satisfies conditions (i) and (ii) above tiles $\mathbf{R}^d$ (with weight 1) by translation by some set $\mathcal{S} \subseteq \mathbf{Z}^d$ if and only if the measure of T is positive. Regarding lattice tilings of T, it is known now [**LaW2**] (and quite hard to prove) that, whenever $\mathcal{D}$ is a *standard* digit set, i.e., whenever $\mathcal{D}$ is a complete set of representatives of the cosets of $\mathbf{Z}^d/A\mathbf{Z}^d$, then there is a lattice $\Gamma \subseteq \mathbf{Z}^d$ such that T tiles with Γ and with weight 1. Thus Theorem 1 can be viewed as an easy version of this last result, one in which we do not necessarily get the weight of the tiling to be 1. However, Theorem 1 applies to much more general objects than indicator functions of compact sets and its proof, which uses Fourier Analysis, is rather straightforward. Even when f is an indicator function, our proof has the advantage that we do not need to worry about the topological properties of T (as in the proof in [**LaW**]) beyond its measurability and finite measure.

§1. Proof of the Theorem

Let

$$\Lambda^* = \left\{x \in \mathbf{R}^d : \ \forall \lambda \in \Lambda \ \langle x, \lambda \rangle \in \mathbf{Z}\right\}$$

be the *dual lattice* of Λ. It turns out that, if $\Lambda = B\mathbf{Z}^d$, for some $B \in M_d(\mathbf{R})$, then $\Lambda^* = B^{-\top}\mathbf{Z}^d$. We shall use the following criterion (Theorem 2) for lattice tiling (for a more detailed proof see, for example, [**K**]).

The Fourier Transform we use in this paper is defined for $f \in L^1(\mathbf{R}^d)$ with the normalization

$$\widehat{f}(\xi) = \int_{\mathbf{R}^d} f(x) \exp(-2\pi i \langle x, \xi \rangle) \, dx.$$

Theorem 2 *Assume that $f \in L^1(\mathbf{R}^d)$. Then f tiles $\mathbf{R}^d$ with a lattice Λ and some weight w if and only if its Fourier Transform, $\widehat{f}$, vanishes on $\Lambda^* \setminus \{0\}$. In this case we have* $w = (\mathrm{vol}\,\Lambda)^{-1} \int_{\mathbf{R}^d} f$.

Proof. This is almost immediate if one notices that the function $\sum_{\lambda \in \Lambda} f(x-\lambda)$ is a function in $L^1(\mathbf{R}^d/\Lambda)$ whose non-constant Fourier coefficients are exactly the values of the Fourier Transform of f at $\Lambda^* \setminus \{0\}$. For the details see [**K**], where more applications of Theorem 2 to lattice tilings of some simple polyhedra can also be found.
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Proof of Theorem 1. Taking the Fourier Transform of both sides of the functional equation (1) we get

$$\widehat{f}(A^\top \xi) = \widehat{f}(\xi) \cdot \varphi(\xi), \tag{3}$$

where

$$\varphi(\xi) = \frac{1}{|\det A|} \sum_{y \in \mathbf{Z}^d} \rho(y) \exp(-2\pi i \langle y, \xi \rangle).$$

Iterating (3) we get

$$\widehat{f}(A^{\top k} \xi) = \widehat{f}(\xi) \prod_{j=0}^{k-1} \varphi(A^{\top j} \xi), \quad \text{for all } k \geq 0. \tag{4}$$

Observe that, since A is expanding, whenever $\xi \neq 0$ we get $A^{\top k}\xi \to \infty$, as $k \to \infty$, and, therefore, by the Riemann-Lebesgue lemma the left hand side of (4) tends to 0. (The fact that we need to use the Riemann-Lebesgue lemma is the reason that we restrict ourselves to L^1 functions and do not consider integral self-affine *measures*.)

Since Λ is A-invariant it follows that Λ^* is $A^\top$-invariant and therefore for each $\xi \in \Lambda^*$ we have $A^{\top j}\xi \in \Lambda^*$ for all $j \geq 0$. It follows that for all $\xi \in \Lambda^*$ and all $j \geq 0$

$$\varphi(A^{\top j}\xi) = \frac{1}{|\det A|} \sum_{y \in \mathbf{Z}^d} \rho(y) = 1,$$

since $\rho(y) \neq 0$ implies $y \in \Lambda$. Thus, for $\xi \in \Lambda^* \setminus \{0\}$, if we take limits in (4) we obtain $\widehat{f}(\xi) = 0$.

By Theorem 2 we get that f tiles $\mathbf{R}^d$ with Λ and weight

$$\begin{aligned} w &= (\operatorname{vol}\Lambda)^{-1} \cdot \int_{\mathbf{R}^d} f(x)\, dx \\ &= \left|\mathbf{Z}^d : \Lambda\right|^{-1} \cdot \int_{\mathbf{R}^d} f(x)\, dx. \end{aligned}$$

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§2. Bibliography

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