Packing, tiling, orthogonality and completeness

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Abstract

Let $\Omega \subseteq \mathbb{R}^d$ be an open set of measure 1. An open set $D \subseteq \mathbb{R}^d$ is called a "tight orthogonal packing region" for Ω if $D - D$ does not intersect the zeros of the Fourier Transform of the indicator function of Ω and D has measure 1. Suppose that Λ is a discrete subset of $\mathbb{R}^d$. The main contribution of this paper is a new way of proving the following result (proved by different methods by Lagarias, Reeds and Wang [9] and, in the case of Ω being the cube, by Iosevich and Pedersen [3]): D tiles $\mathbb{R}^d$ when translated at the locations Λ if and only if the set of exponentials $E_\Lambda = \{\exp 2\pi i\langle\lambda, x\rangle : \ \lambda \in \Lambda\}$ is an orthonormal basis for $L^2(\Omega)$. (When Ω is the unit cube in $\mathbb{R}^d$ then it is a tight orthogonal packing region of itself.) In our approach orthogonality of E_Λ is viewed as a statement about "packing" $\mathbb{R}^d$ with translates of a certain nonnegative function and, additionally, we have completeness of E_Λ in $L^2(\Omega)$ if and only if the above-mentioned packing is in fact a tiling. We then formulate the tiling condition in Fourier Analytic language and use this to prove our result.

§0. Introduction

Notation.
Let $\Omega \subset \mathbb{R}^d$ be measurable of measure 1. The Hilbert space $L^2(\Omega)$ is equipped with the inner product

$$\langle f, g\rangle_\Omega = \int_\Omega f(x)\overline{g(x)}\ dx.$$

Define

$$e_\lambda(x) = \exp 2\pi i\langle\lambda, x\rangle,$$

and, for $\Lambda \subseteq \mathbb{R}^d$,

$$E_\Lambda = \{e_\lambda : \ \lambda \in \Lambda\}.$$

For every continuous function $h : \mathbb{R}^d \to \mathbb{C}$ we write

$$Z(h) = \left\{x \in \mathbb{R}^d : \ h(x) = 0\right\}.$$

Whenever we fail to mention it it should be understood that the measure of $\Omega \subset \mathbb{R}^d$ is equal to 1.

The indicator function of a set E is denoted by $\mathbf{1}_E$.

We denote by $B_r(x)$ the ball in $\mathbb{R}^d$ of radius r centered at x.

When A and B are two sets in $\mathbb{R}^d$ we write $A + B$ for the set of all sums $a + b$, $a \in A$, $b \in B$. Similarly we write $A - B$ for all differences $a - b$, $a \in A$, $b \in B$. For $\lambda \in \mathbb{R}$ we denote by λA the set $\{\lambda a : \ a \in A\}$.

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If O is an open set in $\mathbb{R}^d$ we denote by $C_c^\infty(O)$ the set of all infinitely differentiable functions with support contained in O.

Definition 1 (Spectral sets)
Suppose that Ω is a measurable set of measure 1. We call Ω spectral if $L^2(\Omega)$ has an orthonormal basis $E_\Lambda = \{e_\lambda : \lambda \in \Lambda\}$ of exponentials. The set Λ is then called a spectrum for Ω.

We can always restrict our attention to sets Λ containing 0 and we shall do so without further mention.

Definition 2 (Packing and tiling by nonnegative functions)
(i) A nonnegative measurable function $f : \mathbb{R}^d \to \mathbb{R}^+$ (the "tile") is said to pack a region $S \subseteq \mathbb{R}^d$ with the set $\Lambda \subseteq \mathbb{R}^d$ (the "set of translates") if

$$\sum_{\lambda\in\Lambda} f(x-\lambda) \le 1 \text{ for a.e. } x \in S.$$

In this case we write "$f + \Lambda$ packs S". When S is omitted we understand $S = \mathbb{R}^d$.
(ii) A nonnegative measurable function $f : \mathbb{R}^d \to \mathbb{R}^+$ is said to tile a region $S \subseteq \mathbb{R}^d$ at level ℓ with the set $\Lambda \subseteq \mathbb{R}^d$ if

$$\sum_{\lambda\in\Lambda} f(x-\lambda) = \ell \text{ for a.e. } x \in S.$$

(When not specified $\ell = 1$.) Again we write "$f + \Lambda$ tiles S at level ℓ" (or $f + \Lambda = \ell S$) and $S = \mathbb{R}^d$ is understood when S is omitted.

If $f = \mathbf{1}_E$ is the indicator function of a measurable set E we also say that "$E + \Lambda$" is a packing (resp. tiling) instead of "$\mathbf{1}_E + \Lambda$ is a packing" (resp. tiling).

The following conjecture of Fuglede [1] is still unresolved and has provided the motivation of the problem we deal with in this paper.

Conjecture 1 (Fuglede)
Let Ω be a bounded open set of measure 1. Then Ω is spectral if and only if Ω tiles $\mathbb{R}^d$ by translation.

As an example of a spectral set in $\mathbb{R}^2$ we give the open unit square $(-1/2, 1/2)^2$ which tiles the plane when translated by $\mathbb{Z}^2$ and also has $\mathbb{Z}^2$ as its spectrum. (Note that in Fuglede's conjecture the set of translates by which a tile Ω tiles space need not be the same as its spectrum.) Fuglede [1] proved that the triangle and the disk in the plane are both not spectral. Further in this direction of confirming the conjecture we mention that Iosevich, Katz and Pedersen [2] have recently proved that the ball in $\mathbb{R}^d$ is not spectral and the author [6] proved that any non-symmetric convex domain in $\mathbb{R}^d$ is not spectral. Convex domains that tile space by translation are known to be necessarily symmetric (see [11]).

A related problem is, given a specific set Ω that tiles space by translation, to determine its spectra. Because of its simplicity the cube has been studied the most. Lagarias, Reeds and Wang [9] and Iosevich and Pedersen [3] recently proved that if $Q = (-1/2, 1/2)^d$ is the unit cube in $\mathbb{R}^d$ then $Q + \Lambda$ is a tiling if and only if E_Λ is an orthonormal basis for Q. (We remark here that there exist "exotic" translational tilings by the unit cube which are non-lattice, see [10].) This had been conjectured by Jorgensen and Pedersen [4] where it was proved for dimension $d \le 3$. The purpose of our paper is to give an alternative and, perhaps, more illuminating proof of this fact, which is based on a characterization of translational tiling by a Fourier Analytic criterion.

We follow the terminology of [9].

The two basic tools in this paper are Theorem 2 and Theorem 6. Theorem 2, which is interesting and rather unexpected in itself, concerns tilings by two different tiles and states that if two tiles A and B, of the same volume, both pack with Λ then they either both tile or both do not. This is almost obvious if Λ is periodic but, interestingly, it holds in general and its proof is rather simple. Theorem 6 is the final form for our Fourier Analytic criterion.

Definition 3 (Density)
A set $\Lambda \subseteq \mathbb{R}^d$ has asymptotic density ρ if

$$\lim_{R\to\infty} \frac{\#(\Lambda \cap B_R(x))}{|B_R(x)|} \to \rho$$

uniformly in $x \in \mathbb{R}^d$.
We say that Λ has (uniformly) bounded density if the fraction above is bounded by a constant ρ uniformly for $x \in \mathbb{R}$ and $R > 1$. We say then that Λ has density (uniformly) bounded by ρ.

If $f \geq 0$ then it is clear that if $f + \Lambda$ is a tiling at level $\ell > 0$ then Λ has asymptotic density equal to $\ell / \int f$.

§1. Packing, tiling, orthogonality and completeness

Let Ω be a measurable set in $\mathbb{R}^d$ of measure 1 and Λ be a countable set of points in $\mathbb{R}^d$.

The set $E_\Lambda = \{e_\lambda(x) : \quad \lambda \in \Lambda\}$ is an orthogonal set of exponentials for Ω if and only if

$$\sum_{\lambda\in\Lambda} |\langle e_x, e_\lambda\rangle_\Omega|^2 \leq 1,$$

for each $x \in \mathbb{R}^d$. Since

$$\langle e_x, e_\lambda\rangle_\Omega = \int e^{2\pi i(x-\lambda)t} \mathbf{1}_\Omega(t)\ dt = \widehat{\mathbf{1}_\Omega}(\lambda - x),$$

we conclude that Λ is an orthogonal set for Ω if and only if $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a packing of $\mathbb{R}^d$.

In this case Λ has uniformly bounded density.

Similarly, Λ is a spectrum of Ω (E_Λ is orthogonal and complete) if and only if

$$\sum_{\lambda\in\Lambda} |\langle e_x, e_\lambda\rangle_\Omega|^2 = 1,$$

for all $x \in \mathbb{R}^d$. That is, Λ is a spectrum of Ω if and only if $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a tiling of $\mathbb{R}^d$.

Definition 4 The open set D is called an orthogonal packing region for Ω if

$$(D - D) \cap Z(\widehat{\mathbf{1}_\Omega}) = \emptyset.$$

By the definition of an orthogonal packing region D for Ω, if Λ is an orthogonal set of exponentials for Ω then $D + \Lambda$ is a packing of $\mathbb{R}^d$. Indeed, if $\lambda, \mu \in \Lambda$, $\lambda \neq \mu$, then $\widehat{\mathbf{1}_\Omega}(\lambda - \mu) = 0$, since $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a packing, which implies $\lambda - \mu \in Z(\widehat{\mathbf{1}_\Omega})$ which is disjoint from $D - D$. Hence $(\Lambda - \Lambda) \cap (D - D) = \{0\}$ and $D + \Lambda$ is a packing.

We summarize these observations in the following theorem.

Theorem 1 *Let Ω be a measurable set in $\mathbb{R}^d$ of measure 1 and $\Lambda \subset \mathbb{R}^d$ be countable.*

1. *E_Λ is an orthogonal set for $L^2(\Omega)$ if and only if $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a packing.*
2. *E_Λ is an orthonormal basis for $L^2(\Omega)$ (a spectrum for Ω) if and only if $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a tiling.*
3. *If D is an orthogonal packing region for Ω and E_Λ is an orthogonal set in $L^2(\Omega)$ then $D + \Lambda$ is a packing.*

§2. A result about packing and tiling by two different tiles

The following theorem is a crucial tool for the results of this paper. It is intuitively clear when Λ is a periodic set but it is, perhaps, suprising that it holds without any assumptions on the set Λ. Its proof is very simple.

Theorem 2 *If $f, g \geq 0$, $\int f(x)dx = \int g(x)dx = 1$ and both $f + \Lambda$ and $g + \Lambda$ are packings of $\mathbb{R}^d$, then $f + \Lambda$ is a tiling if and only if $g + \Lambda$ is a tiling.*

Proof. We first show that, under the assumptions of the Theorem,

$$f + \Lambda \text{ tiles } -\operatorname{supp} g \implies g + \Lambda \text{ tiles } -\operatorname{supp} f. \tag{1}$$

Indeed, if $f + \Lambda$ tiles $-\operatorname{supp} g$ then

$$1 = \int g(-x) \sum_{\lambda\in\Lambda} f(x-\lambda)\ dx = \sum_{\lambda\in\Lambda} \int g(-x) f(x-\lambda)\ dx,$$

which, after the change of variable $y = -x + \lambda$, gives

$$1 = \int f(-y) \sum_{\lambda\in\Lambda} g(y-\lambda)\ dy.$$

This in turn implies, since $\sum_{\lambda\in\Lambda} g(y-\lambda) \leq 1$, that $\sum_\lambda g(y-\lambda) = 1$ for a.e. $y \in -\operatorname{supp} f$.

To complete the proof of the theorem, notice that if $f+\Lambda$ is a tiling of $\mathbb{R}^d$ and $a \in \mathbb{R}^d$ is arbitrary then both $f(x-a)+\Lambda$ and $g(x-a)+\Lambda$ are packings and $f+\Lambda$ tiles $-\operatorname{supp} g(x-a) = -\operatorname{supp} g - a$. We conclude that $g(x-a)+\Lambda$ tiles $-\operatorname{supp} f$, or $g+\Lambda$ tiles $-\operatorname{supp} f - a$. Since $a \in \mathbb{R}^d$ is arbitrary we conclude that $g + \Lambda$ tiles $\mathbb{R}^d$.
■

§3. Fourier Analytic criteria for tiling

The action of a tempered distribution (see [12]) α on a Schwartz function ϕ is denoted by $\alpha(\phi)$. The Fourier Transform of α is defined by the equation

$$\widehat{\alpha}(\phi) = \alpha(\widehat{\phi}).$$

The support $\operatorname{supp}\alpha$ is the smallest closed set F such that for any smooth ϕ of compact support contained in the open set F^c we have $\alpha(\phi) = 0$.

If $f \in L^1(\mathbb{R}^d)$ and $\widehat{f}$ is C^∞ then, if $\Lambda \subset \mathbb{R}^d$ is a discrete set, the following theorem, first proved in [8] in dimension 1, gives necessary and sufficient conditions for $f + \Lambda$ to be a tiling. We give the proof here for completeness.

Theorem 3 *Suppose $f \in L^1(\mathbb{R}^d)$ and $\widehat{f} \in C^\infty$. Suppose also that Λ is a discrete subset of $\mathbb{R}^d$ of bounded density. Write δ_Λ for the tempered distribution $\sum_{\lambda\in\Lambda}\delta_\lambda$ and $\widehat{\delta_\Lambda}$ for its Fourier Transform.*
(i) If $f+\Lambda$ is a tiling then

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \left\{\widehat{f}=0\right\} \cup \{0\}. \tag{2}$$

(ii) If $\widehat{\delta_\Lambda}$ is locally a measure then (2) *implies that $f+\Lambda$ is a tiling.*

Notice that whenever f has compact support the function $\widehat{f}$ is smooth.

Proof of Theorem 3. (i) If $f+\Lambda$ is a tiling then $1 = f * \delta_\Lambda$, hence, taking Fourier Transforms, $\delta_0 = \widehat{f}\widehat{\delta_\Lambda}$. Take $\phi \in C_c^\infty(\mathbb{R}^d \setminus K)$, where

$$K = \left\{\widehat{f}=0\right\} \cup \{0\}.$$

Then

$$\widehat{\delta_\Lambda}(\phi) = (\widehat{f}\widehat{\delta_\Lambda})\left(\frac{\phi}{\widehat{f}}\right) = \delta_0\left(\frac{\phi}{\widehat{f}}\right) = \frac{\phi}{\widehat{f}}(0) = 0.$$

This proves (2). Note that it was crucial in the proof that $\dfrac{\phi}{\widehat{f}}$ is a function in $C_c^\infty(\mathbb{R}^d \setminus K)$ and this is why we demanded that $\widehat{f}$ is smooth.

(ii) Take ϕ to be a Schwartz function. We have

$$(f * \delta_\Lambda)(\phi) = (\widehat{f}\widehat{\delta_\Lambda})(\widehat{\phi}) = \widehat{\delta_\Lambda}(\widehat{\phi}\widehat{f}).$$

However, this is

$$\widehat{\phi}(0)\widehat{f}(0)\widehat{\delta_\Lambda}(\{0\}),$$

as $\widehat{\delta_\Lambda}$, being a measure, kills any continuous function vanishing on $\operatorname{supp}\widehat{\delta_\Lambda}$. Since ϕ is arbitrary we conclude that $f * \delta_\Lambda$ is a constant.
■

We shall need a different version of Theorem 3 here. In the theorem that follows compact support and nonnegativity of $\widehat{f}$ compensate for its lack of smoothness. This theorem has also been proved and used by the author in [6].

Theorem 4 *Suppose that $f \ge 0$ is not identically 0, that $f \in L^1(\mathbb{R}^d)$, $\widehat{f} \ge 0$ has compact support and $\Lambda \subset \mathbb{R}^d$. If $f+\Lambda$ is a tiling then*

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq \left\{\widehat{f}=0\right\} \cup \{0\}. \tag{3}$$

Proof of Theorem 4. Assume that $f + \Lambda = w\mathbb{R}^d$ and let

$$K = \left\{\widehat{f}=0\right\} \cup \{0\}.$$

We have to show that

$$\widehat{\delta_\Lambda}(\phi) = 0, \quad \forall \phi \in C_c^\infty(K^c).$$

Since $\widehat{\delta_\Lambda}(\phi) = \delta_\Lambda(\widehat{\phi})$ this is equivalent to $\sum_{\lambda\in\Lambda}\widehat{\phi}(\lambda) = 0$, for each such ϕ. Notice that $h = \phi/\widehat{f}$ is a continuous function, but not necessarily smooth. We shall need that $\widehat{h} \in L^1$. This is a consequence of a well-known theorem of Wiener [12, Ch. 11]. We denote by $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ the d-dimensional torus.

Theorem (Wiener)
If $g \in C(\mathbb{T}^d)$ has an absolutely convergent Fourier series

$$g(x) = \sum_{n\in\mathbb{Z}^d} \widehat{g}(n) e^{2\pi i\langle n,x\rangle}, \quad \widehat{g} \in \ell^1(\mathbb{Z}^d),$$

and if g does not vanish anywhere on $\mathbb{T}^d$ then $1/g$ also has an absolutely convergent Fourier series.

Assume that

$$\operatorname{supp}\phi,\ \operatorname{supp}\widehat{f} \subseteq \left(-\frac{L}{2}, \frac{L}{2}\right)^d.$$

Define the function F to be:
(i) periodic in $\mathbb{R}^d$ with period lattice $(L\mathbb{Z})^d$,
(ii) to agree with $\widehat{f}$ on $\operatorname{supp}\phi$,
(iii)to be non-zero everywhere and,
(iv) to have $\widehat{F} \in \ell^1(\mathbb{Z}^d)$, i.e.,

$$\widehat{F} = \sum_{n\in\mathbb{Z}^d} \widehat{F}(n)\delta_{L^{-1}n},$$

is a finite measure in $\mathbb{R}^d$.

One way to define such an F is as follows. First, define the $(L\mathbb{Z})^d$-periodic function $g \ge 0$ to be $\widehat{f}$ periodically extended. The Fourier coefficients of g are $\widehat{g}(n) = L^{-d} f(-n/L) \ge 0$. Since $g, \widehat{g} \ge 0$ and g is continuous at 0 it is easy to prove that $\sum_{n\in\mathbb{Z}^d} \widehat{g}(n) = g(0)$, and therefore that g has an absolutely convergent Fourier series.

Let ϵ be small enough to guarantee that $\widehat{f}$ (and hence g) does not vanish on $(\operatorname{supp}\phi) + B_\epsilon(0)$. Let k be a smooth $(L\mathbb{Z})^d$-periodic function which is equal to 1 on $(\operatorname{supp}\phi) + (L\mathbb{Z}^d)$ and equal to 0 off $(\operatorname{supp}\phi + B_\epsilon(0)) + (L\mathbb{Z}^d)$, and satisfies $0 \le k \le 1$ everywhere. Finally, define

$$F = kg + (1-k).$$

Since both k and g have absolutely summable Fourier series and this property is preserved under both sums and products, it follows that F also has an absolutely summable Fourier series. And by the nonnegativity of g we get that F is never 0, since $k = 0$ on $Z(\widehat{f}) + (L\mathbb{Z}^d)$.

By Wiener's theorem, $\widehat{F^{-1}} \in \ell^1(\mathbb{Z}^d)$, i.e., $\widehat{F^{-1}}$ is a finite measure on $\mathbb{R}^d$. We now have that

$$\left(\frac{\phi}{\widehat{f}}\right)^\wedge = \widehat{\phi F^{-1}} = \widehat{\phi} * \widehat{F^{-1}} \in L^1(\mathbb{R}^d).$$

This justifies the interchange of the summation and integration below:

$$\begin{aligned}
\sum_{\lambda\in\Lambda} \widehat{\phi}(\lambda) &= \sum_{\lambda\in\Lambda} \left(\frac{\phi}{\widehat{f}}\widehat{f}\right)^\wedge (\lambda) \\
&= \sum_{\lambda\in\Lambda} \left(\frac{\phi}{\widehat{f}}\right)^\wedge * \widehat{\widehat{f}}\,(\lambda) \\
&= \sum_{\lambda\in\Lambda} \int_{\mathbb{R}^d} \left(\frac{\phi}{\widehat{f}}\right)^\wedge (y) f(y-\lambda)\, dy \\
&= \int_{\mathbb{R}^d} \left(\frac{\phi}{\widehat{f}}\right)^\wedge (y) \sum_{\lambda\in\Lambda} f(y-\lambda)\, dy
\end{aligned}$$

$$
\begin{aligned}
&= w \int_{\mathbb{R}^d} \left(\frac{\phi}{\widehat{f}}\right)^{\wedge} (y)\, dy \\
&= w \frac{\phi}{\widehat{f}}(0) \\
&= 0,
\end{aligned}
$$

as we had to show.

■

In the other direction assume that we have

$$\operatorname{supp} \widehat{\delta_\Lambda} \subseteq \left\{\widehat{f} = 0\right\} \cup \{0\} \tag{4}$$

for some non-zero $f \geq 0$ in L^1 and that Λ is of bounded density. Since $\widehat{f}(0) = \int f > 0$ it follows that in some neighborhood N of 0 we have $(\operatorname{supp} \widehat{\delta_\Lambda}) \cap N = \{0\}$. Hence the set

$$O = \left(\operatorname{supp} \widehat{\delta_\Lambda} \setminus \{0\}\right)^c \tag{5}$$

is open and

$$\left\{\widehat{f} \neq 0\right\} \subseteq O.$$

Theorem 5 *Suppose that* $0 \leq f \in L^1(\mathbb{R}^d)$, $\int f = 1$, Λ *(of uniformly bounded density) is of density* 1, *and that* (4) *holds. Suppose also that for the open set* O *of* (5) *and for each* $\epsilon > 0$ *there exists* $f_\epsilon \geq 0$ *in* $L^1(\mathbb{R}^d)$ *such that* $\widehat{f_\epsilon}$ *is in* C^∞, $\operatorname{supp} \widehat{f_\epsilon} \subseteq O$ *and*

$$\|f - f_\epsilon\|_{L^1} \leq \epsilon.$$

Then $f + \Lambda$ *is a tiling.*

Example. All bounded open convex sets O have the property required by the theorem, for all functions $f \geq 0$ such that $\widehat{f}$ is non-zero only in O. To see this, assume, without loss of generality, that $0 \in O$. Construct then the functions ($\epsilon \to 0$)

$$\widehat{f_\epsilon}(x) = \psi_\epsilon(x) * \widehat{f}\left(\frac{x}{1-\epsilon}\right),$$

where ψ_ϵ is a smooth, positive-definite approximate identity supported in $(\epsilon/2)O$. Then $\widehat{f_\epsilon}$ is smooth, supported properly in O and f_ϵ converges to f in L^1 (for example, by the dominated convergence theorem). We will not need the above observation about convex domains below.

Proof of Theorem 5. Suppose that f_ϵ is as in the Theorem. First we show that $(\int f_\epsilon)^{-1} f_\epsilon + \Lambda$ is a tiling. That is, we show that the convolution $f_\epsilon * \delta_\Lambda$ is a constant. Let ϕ be C_c^∞ function. Then

$$(f_\epsilon * \delta_\Lambda)(\phi) = \widehat{f_\epsilon}\widehat{\delta_\Lambda}(\widehat{\phi}) = \widehat{\delta_\Lambda}(\widehat{\phi}\widehat{f_\epsilon}).$$

But the function $\widehat{\psi} = \widehat{\phi}\widehat{f_\epsilon}$ is a C_c^∞ function whose support intersects $\operatorname{supp} \widehat{\delta_\Lambda}$ only at 0. And, it is not hard to show, because Λ has density 1, that $\widehat{\delta_\Lambda}$ is equal to δ_0 in a neighborhood of 0 (see [7]). Hence

$$(f_\epsilon * \delta_\Lambda)(\phi) = \left(\widehat{\phi}\widehat{f_\epsilon}\right)(0) = \int \phi \int f_\epsilon,$$

and, since this is true for an arbitrary C_c^∞ function ϕ, we conclude that $f_\epsilon * \delta_\Lambda = \int f_\epsilon$, as we had to show.

For any set Λ of uniformly bounded density we have (B is any ball in $\mathbb{R}^d$ and $g \in L^1(\mathbb{R}^d)$)

$$\int_B \left| \sum_{\lambda\in\Lambda} g(x-\lambda) \right| dx \le C_{B,\Lambda} \int_{\mathbb{R}^d} |g|,$$

(See [8] for a proof of this in dimension 1, which holds for any dimension.) Applying this for $g = f - f_\epsilon$ we obtain that

$$\sum_{\lambda\in\Lambda} f_\epsilon(x-\lambda) \to \sum_{\lambda\in\Lambda} f(x-\lambda), \quad \text{in } L^1(B).$$

Since B is arbitrary this implies that $\sum_{\lambda\in\Lambda} f(x-\lambda) = 1$, a.e. in $\mathbb{R}^d$.
■

We write $\widetilde{f}(x) = \overline{f(-x)}$.

Let $\Omega \subset \mathbb{R}^d$ be a bounded open set of measure 1, $\mathbf{1}_\Omega$ its indicator function and f be such that $\widehat{f} = \mathbf{1}_\Omega * \widetilde{\mathbf{1}_\Omega}$. Then $\widetilde{f} = \left|\widehat{\mathbf{1}_\Omega}\right|^2 \ge 0$, $\int f = 1$ by Parseval's theorem. Clearly we have $\left\{\widehat{f} \ne 0\right\} = \Omega - \Omega$.

Write

$$\Omega_\epsilon = \{x \in \Omega : \ \mathrm{dist}\,(x, \partial\Omega) > \epsilon\},$$

and define f_ϵ by

$$\widehat{f_\epsilon} = \psi_\epsilon * \mathbf{1}_{\Omega_\epsilon} * (\psi_\epsilon * \mathbf{1}_{\Omega_\epsilon})^{\widetilde{}}$$

(or $\widetilde{f_\epsilon} = \left|\widehat{\psi_\epsilon}\right|^2 \left|\widehat{\mathbf{1}_{\Omega_\epsilon}}\right|^2$), where ψ_ϵ is a smooth, positive-definite approximate identity supported in $B_{\epsilon/2}(0)$.

One can easily prove the following proposition.

If $g_n \to g$ in L^2 then $|g_n|^2 \to |g|^2$ in L^1.

(For the proof just notice the identity

$$|g|^2 - |g_n|^2 = |g - g_n|^2 + 2 \cdot \mathrm{Re}\left(\overline{g_n}(g - g_n)\right),$$

integrate and use the triangle and Cauchy-Schwartz inequalities.)

Since $\psi_\epsilon * \mathbf{1}_{\Omega_\epsilon} \to \mathbf{1}_\Omega$ in L^2 (dominated convergence) we have (Parseval) that $\widehat{\psi_\epsilon}\widehat{\mathbf{1}_{\Omega_\epsilon}} \to \widehat{\mathbf{1}_\Omega}$ in L^2 and, using the proposition above, that $\left|\widehat{\psi_\epsilon}\right|^2 \left|\widehat{\mathbf{1}_{\Omega_\epsilon}}\right|^2 \to \left|\widehat{\mathbf{1}_\Omega}\right|^2$ in L^1, which means that $f_\epsilon \to f$ in L^1.

We also have that

$$\mathrm{supp}\, \widehat{f_\epsilon} \subseteq \overline{\Omega_{\epsilon/2}} - \overline{\Omega_{\epsilon/2}} \subseteq \Omega - \Omega = \left\{\widehat{f} \ne 0\right\}.$$

The assumptions of Theorem 5 are therefore satisfied. Combining Theorems 4 and 5 with the above observations we obtain the following characterization of tiling which we will use throughout the rest of the paper.

Theorem 6 *Let Ω be a bounded open set, Λ a discrete set in $\mathbb{R}^d$, and $\delta_\Lambda = \sum_{\lambda\in\Lambda} \delta_\lambda$. Then $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a tiling if and only if Λ has uniformly bounded density and*

$$(\Omega - \Omega) \cap \mathrm{supp}\, \widehat{\delta_\Lambda} = \{0\}.$$

§4. Size of orthogonal packing regions. Spectra of the cube.

The following theorem was conjectured in [9] (Conjecture 2.1).

Theorem 7 *If Ω has measure 1 and tiles $\mathbb{R}^d$ then $|D| \le 1$ for any orthogonal packing region D of Ω.*

Proof of Theorem 7. Assume that $\Omega + \Lambda$ is a tiling. Then $\operatorname{dens}\Lambda = 1$. By Theorem 3

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq Z(\widehat{\mathbf{1}_\Omega}) \cup \{0\}.$$

Since D is an orthogonal packing region for Ω we have by definition, and since $D - D = \left\{\mathbf{1}_D * \widetilde{\mathbf{1}_D} \neq 0\right\}$,

$$Z(\widehat{\mathbf{1}_\Omega}) \subseteq Z(\mathbf{1}_D * \widetilde{\mathbf{1}_D}).$$

Therefore

$$\operatorname{supp}\widehat{\delta_\Lambda} \subseteq Z(\mathbf{1}_D * \widetilde{\mathbf{1}_D}) \cup \{0\},$$

and by Theorem 6 we obtain that $\left|\widehat{\mathbf{1}_D}\right|^2 + \Lambda$ is a tiling at level

$$\operatorname{dens}\Lambda \int \left|\widehat{\mathbf{1}_D}\right|^2 = |D| \quad \text{(Parseval)}.$$

On the other hand, the level of the tiling is (evaluating at 0)

$$\sum_\lambda \left|\widehat{\mathbf{1}_D}\right|^2(-\lambda) \ge \left|\widehat{\mathbf{1}_D}\right|^2(0) = \int \mathbf{1}_D * \widetilde{\mathbf{1}_D} = |D|^2.$$

Hence $|D| \ge |D|^2$ or $|D| \le 1$.
■

Definition 5 (Tight orthogonal packing regions, tight spectral pairs)
The open set D is called a tight orthogonal packing region for Ω if it is an orthogonal packing region for Ω and $|D| = 1$.

A pair (Ω, D) of bounded open sets in $\mathbb{R}^d$ is called a tight spectral pair if each is a tight orthogonal packing region for the other.

The following result has also been proved in [9] (Theorem 3.1) but for a smaller class of admissible open sets Ω.

Theorem 8 *Suppose Ω, Ω' are bounded open sets in $\mathbb{R}^d$ of measure 1. Suppose also that Λ is an orthogonal set of exponentials for Ω and that $\Omega' + \Lambda$ is a packing.*

Then Λ is a spectrum for Ω if and only if $\Omega' + \Lambda$ is a tiling.

Proof. Since E_Λ is an orthogonal set in $L^2(\Omega)$ it follows that $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a packing and Λ is a spectrum for Ω if and only if $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is also a tiling. Notice that

$$\int \left|\widehat{\mathbf{1}_\Omega}\right|^2 = \int \mathbf{1}_{\Omega'} = 1.$$

By Theorem 2 it follows that $\Omega' + \Lambda$ is a tiling if and only if $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a tiling, as we had to show.
■

The following theorem relates, for a tight spectral pair (Ω, D), the tilings of D with the spectra of Ω.

Theorem 9 *Assume that (Ω, D) is a tight spectral pair. Then Λ is a spectrum of Ω if and only if $D + \Lambda$ is a tiling.*

Proof. Λ is a spectrum of Ω if and only if $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a tiling (Theorem 1).

"$\Longleftarrow$" If $D + \Lambda$ is a tiling then $\operatorname{supp}\widehat{\delta_\Lambda} \subseteq Z(\widehat{\mathbf{1}_D}) \cup \{0\}$ (by Theorem 3), which is a subset of $Z(\mathbf{1}_\Omega * \widetilde{\mathbf{1}_\Omega}) \cup \{0\}$ (this is because Ω is an orthogonal packing region for D). Hence $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a tiling (Theorem 6).

"$\Longrightarrow$" In the other direction, suppose that $\left|\widehat{\mathbf{1}_\Omega}\right|^2 + \Lambda$ is a tiling. Then Λ is an orthogonal set for Ω and hence $D + \Lambda$ is a packing because D is an orthogonal packing region for Ω (Theorem 1(3)). But Theorem 2 implies then that $D + \Lambda$ is a tiling as well.
■

Let $Q = (-1/2, 1/2)^d$ be the 0-centered unit cube in $\mathbb{R}^d$. An easy calculation gives for the Fourier transform of $\mathbf{1}_Q$:

$$\widehat{\mathbf{1}_Q}(\xi_1, \ldots, \xi_d) = \prod_{j=1}^{d} \frac{\sin \pi\xi_j}{\pi\xi_j}. \tag{6}$$

It follows that $\widehat{\mathbf{1}_Q}$ vanishes precisely at those points with at least one non-zero integer coordinate. From the definition of an orthogonal packing region it follows that (Q, Q) is a tight spectral pair. Hence we have the following, which has already been proved in [9, 3].

Corollary 1 *Λ is a spectrum of Q if and only if $Q + \Lambda$ is a tiling.*

Let us also mention another case, in dimension 1, when Theorem 9 applies: the set $\Omega = (0, 1/2) \cup (1, 3/2)$ has itself as a tight orthogonal packing region, hence all its spectra are also tiling sets for Ω and vice versa (see [9]).

§5. Generalization of a theorem of Keller

In [9, 3] the following old result of Keller [5] was used in order to prove that the spectra of the unit cube Q of $\mathbb{R}^d$ are exactly those sets Λ such that $Q + \Lambda$ is tiling (our Corollary 1).

Theorem (Keller)
If $Q + \Lambda$ is a tiling and $0 \in \Lambda$ then each non-zero $\lambda \in \Lambda$ has a non-zero integer coordinate.

The use of Keller's theorem is avoided in our approach. Furthermore it is an easy consequence of what we have already proved.

Having computed $\widehat{\mathbf{1}_Q}$ and its zero-set in the previous paragraph it is now evident that Keller's theorem is a special case of the following result.

Theorem 10 *Assume that (Ω, D) is a tight spectral pair and that $\Omega + \Lambda$ is a tiling ($0 \in \Lambda$). Then*

$$\Lambda \setminus \{0\} \subseteq Z(\widehat{\mathbf{1}_D}).$$

Proof of Theorem 10. Since $\Omega + \Lambda$ is a tiling we obtain that $\left|\widehat{\mathbf{1}_D}\right|^2 + \Lambda$ is a tiling (i.e., Λ is a spectral set for D) and, since $\left|\widehat{\mathbf{1}_D}\right|^2(0) = 1$, we obtain

$$\widehat{\mathbf{1}_D}(\lambda) = 0, \quad (\lambda \in \Lambda \setminus \{0\}).$$

■

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