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# A PROBLEM OF STEINHAUS: CAN ALL PLACEMENTS OF A PLANAR SET CONTAIN EXACTLY ONE LATTICE POINT?

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*Dedicated to Heini Halberstam*

ABSTRACT. STEINHAUS asked whether there exists a subset of the plane which, no matter how translated and rotated, always contains exactly one point with integer coordinates. The best result on this problem so far has been that of BECK who showed, using harmonic analysis, that no such bounded measurable set exists.

By simplifying BECK's method and by pushing it to its limits we improve his result. We show that if a set is measurable and there exists a direction such that the set is very small outside large strips parallel to the direction, then the set cannot have the property of STEINHAUS.

## 1. INTRODUCTION

STEINHAUS [4, problem 59] asked if there exists a subset of the plane which, no matter how translated and rotated, always contains *exactly one* lattice point, that is a point with integer coordinates. Let us say that such a set satisfies the STEINHAUS *property.*

**Definition 1.** A set $S \subseteq \mathbf{R}^2$ has the STEINHAUS property if for every $x \in \mathbf{R}^2$ and for every rotation $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ we have

$$\#\left(\mathbf{Z}^2 \cap (AS + x)\right) = 1, \tag{1}$$

where $AS + x = \{As + x : \; s \in S\}$.

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SIERPIŃSKI [5] proved by elementary methods that no set which is bounded and either closed or open can have the STEINHAUS property.

The best result on this problem[1] is due to BECK [1] who, using harmonic analysis, proved that bounded measurable sets cannot have the STEINHAUS property.

The purpose of this note is to simplify BECK's argument and push the harmonic analysis method to its limits. We improve BECK's result and show that any measurable set whose area outside any strip $\{x \in \mathbf{R}^2 : |\langle x, u\rangle| < R\}$, for some fixed orientation $u \in \mathbf{R}^2$, drops very fast as $R \to \infty$ cannot have the STEINHAUS property (see Theorem 1 below).

We need the following definition.

**Definition 2.** Let $f \in L^1(\mathbf{R}^2)$ and $x, u$ be a pair of orthogonal unit vectors in $\mathbf{R}^2$. The projection of $f$ along $u$ is the function (defined for almost all $t \in \mathbf{R}$)

$$(\Pi_u f)(t) \;=\; \int_{\mathbf{R}} f(tu + sx)\, ds. \tag{2}$$

Notice that $\Pi_u f \in L^1(\mathbf{R})$ by Fubini's theorem.

Our main result is the following.

**Theorem 1.** *Let $S \subseteq \mathbf{R}^2$ be a measurable set for which there exists a direction $u$ (a unit vector in $\mathbf{R}^2$) such that*

$$F(x) \le \exp(-Kx^2 \log^{1/2} x), \text{ for large } x, \tag{3}$$

*where $K > 0$ is a sufficiently large absolute constant and*

$$F(x) \;=\; \int_x^\infty [(\Pi_u \mathbf{1}_S)(t) + (\Pi_u \mathbf{1}_S)(-t)]\, dt$$

*($\mathbf{1}_S$ is the indicator function of $S$).*

*Then $S$ cannot have the STEINHAUS property.*

Observe that $F(x)$ above is the Lebesgue measure of the set

$$S \cap \{v \in \mathbf{R}^2 : |\langle v, u\rangle| > x\}.$$

The proof of Theorem 1 is based on the observation that, if $S$ has the STEINHAUS property, then the Fourier transform of its indicator function must have a very rich set of zeros (Lemma 1). Standard methods involving Jensen's formula are then used to show that the postulated fast decrease of the set outside large strips is incompatible with such a large number of zeros.

[1] I learned very late that the same result had also been proved by H.T. Croft (Quart. J. Math. Oxford Ser. (2) 33 (1982), 71-83) who used a completely different method.

## 2. The Main Lemma

Denoting the indicator function of a set $E$ by $\mathbf{1}_E$, the STEINHAUS property implies

$$\sum_{m,n\in\mathbf{Z}} \mathbf{1}_{AS+x}(m,n) \;=\; 1, \tag{4}$$

for all $x \in \mathbf{R}^2$ and rotations $A$. In what follows we only assume that $S$ satisfies (4) for almost all $x$ (in the sense of Lebesgue measure in $\mathbf{R}^2$) and for a dense set of rotations $A$.

Our main lemma is the following.

**Lemma 1.** *Let $S$ be measurable and assume that it satisfies (4) for almost all $x \in \mathbf{R}^2$ and for $A \in \mathcal{A}$, where $\mathcal{A}$ is a dense set of rotations. Then the Fourier transform $\widehat{\mathbf{1}_S}$ of the indicator function $\mathbf{1}_S$ vanishes on all circles centered at the origin that contain a lattice point.*

**Remarks:**

**(a)** Our definition of the Fourier transform $\widehat{f}$ of a function $f \in L^1(\mathbf{R}^2)$ is

$$\widehat{f}(\xi,\eta) \;=\; \int_{\mathbf{R}^2} e^{-2\pi i(\xi x+\eta y)} f(x,y)\; dxdy.$$

**(b)** Note that for any $f \in L^1(\mathbf{R}^2)$ and any non-singular $2\times 2$ matrix $A$ we have

$$\widehat{g}(x) \;=\; \frac{1}{|\det A|}\widehat{f}(A^{-t}x), \quad \text{for all } x \in \mathbf{R}^2, \tag{5}$$

for $g(x) = f(Ax)$, where $A^{-t}$ is the inverse of the transpose $A^t$ of the matrix $A$.

**(c)** It is easy to see [1] that if $S$ has the STEINHAUS property and is measurable then it necessarily has measure 1.

We can now proceed with the proof of the main lemma.

*Proof of Lemma* 1. Define

$$f_A(x) = \sum_{y\in\mathbf{Z}^2} \mathbf{1}_S(A^{-1}(y-x)).$$

Notice that, since the measure of $S$ is 1, $f_A \in L^1([0,1]^2)$ and is periodic with $\mathbf{Z}^2$ as the period lattice. We view $f_A$ as a function on the two-dimensional torus.

Assumption (4) implies that for all $A \in \mathcal{A}$ the function $f_A$ is equal to 1 almost everywhere. Therefore all its non-constant Fourier coefficients vanish. On the other hand, for all $\xi \in \mathbf{Z}^2$

$$\widehat{f_A}(\xi) = \widehat{\mathbf{1}_S}(-A^t\xi),$$

by a simple interchange of summation and integration justified by Fubini's theorem. Therefore, if $\xi \neq 0$, we have

$$\widehat{\mathbf{1}_S}(A^t\xi) = 0,$$

and since this holds for all $A \in \mathcal{A}$, $\widehat{\mathbf{1}_S}$ is continuous and $\mathcal{A}$ is dense the proof is complete. □

## 3. Proof of the Main Theorem

We shall need the following two lemmas concerning the analyticity and the number of zeros of the Fourier transform of functions $f : \mathbf{R} \to \mathbf{C}$, now considered as a function of a complex variable:

$$\widehat{f}(z) \;=\; \int_{\mathbf{R}} e^{-2\pi i z x} f(x)\, dx, \quad z \in \mathbf{C}.$$

Lemma 2 is a simple application of Fubini's theorem and Lemma 3 of Jensen's formula [3, p. 82].

**Lemma 2.** *If* $\int_{\mathbf{R}} |f(x)| e^{M|x|}\, dx < \infty$ *for all* $M > 0$ *then* $\widehat{f}(z)$ *is an entire function of* $z$.

*Proof.* For any contour $\Gamma \subseteq \mathbf{C}$ we have

$$\begin{aligned}\oint_\Gamma \widehat{f}(z)\, dz &= \oint_\Gamma \int_{\mathbf{R}} e^{-2\pi i z x} f(x)\, dx\, dz \\ &= \int_{\mathbf{R}} f(x) \left( \oint_\Gamma e^{-2\pi i z x}\, dz \right) dx,\end{aligned}$$

the change of the order of integration justified by Fubini's theorem and our assumption. But the inner contour integral is 0 for all $x$, since the integrand is entire for all $x$, therefore $\oint_\Gamma \widehat{f}(z)\, dz = 0$ for all $\Gamma$. Morera's theorem implies that $\widehat{f}(z)$ is entire. □

**Lemma 3.** *Let* $f \in L^1(\mathbf{R}) \cap C(\mathbf{R})$ *and define* $F(x) = \int_x^\infty (|f(t)| + |f(-t)|)\, dt$. *Assume that*

$$F(x) \le \exp(-K x^2 \log^{1/2} x), \text{ for large } x, \tag{6}$$

*where* $K > 0$ *is a constant. Then*

$$\#\{\xi \in \mathbf{C} :\ \widehat{f}(\xi) = 0 \;\&\; |\xi| \le R\} \;\le\; C(K) \frac{R^2}{\log^{1/2} R}, \tag{7}$$

*where* $C(K) > 0$ *is a function of* $K$ *with* $C(K) \to 0$ *as* $K \to \infty$.

*Proof.* Integration by parts shows that the assumption of Lemma 2 clearly holds for $f$, therefore $\widehat{f}$ is an entire function. We may clearly assume $\widehat{f}(0) = 1$. Jensen's formula is then

$$\sum_{k=1}^{n} \log \frac{R}{r_k} = \frac{1}{2\pi} \int_0^{2\pi} \log \left| \widehat{f}(Re^{i\theta}) \right| \, d\theta,$$

where $r_k$, $k = 1, \ldots, n$, are the moduli of the zeros of $\widehat{f}(z)$ in the disk $\{|z| \le R\}$. We have

$$\begin{aligned} \#\{k : \; r_k \le R/e\} &\le \sum_{r_k \le R/e} \log \frac{R}{r_k} \\ &\le \sum_{k=1}^{n} \log \frac{R}{r_k} \\ &= \frac{1}{2\pi} \int_0^{2\pi} \log \left| \widehat{f}(Re^{i\theta}) \right| \, d\theta. \end{aligned}$$

It suffices to show

$$\log \left| \widehat{f}(Re^{i\theta}) \right| \le C(K) R^2 \log^{-1/2} R, \quad \text{for all large } R. \tag{8}$$

We have

$$\begin{aligned} \left| \widehat{f}(Re^{i\theta}) \right| &= \left| \int_{\mathbf{R}} \exp\left(-2\pi i R e^{i\theta} x\right) f(x) \, dx \right| \\ &\le \int_{\mathbf{R}} e^{2\pi R |x|} |f(x)| \, dx \\ &= \int_0^\infty e^{2\pi R x} (-F'(x)) \, dx \quad \text{(since } f \text{ is continuous)} \\ &= 2\pi R \int_0^\infty e^{2\pi R x} F(x) \, dx - e^{2\pi R x} F(x) \Big|_0^\infty \\ &\le 2\pi R \int_0^\infty e^{2\pi R x} F(x) \, dx + F(0). \qquad (9) \end{aligned}$$

Write $\varphi(x) = K x \log^{1/2} x$ and notice that

$$\begin{aligned} \int_0^\infty e^{2\pi R x} F(x) \, dx &\le \int_0^\infty e^{(2\pi R - \varphi(x)) x} \, dx + O_K(1) \\ &= \int_0^{\varphi^{-1}(4\pi R)} e^{2\pi R x} \, dx + O_K(1), \end{aligned}$$

where $O_K(1)$ is a quantity bounded by a function of $K$ alone. An easy calculation shows that

$$\varphi^{-1}(4\pi R) \le C(K) \frac{R}{\log^{1/2} R},$$

with $C(K)$ decreasing to 0 as $K$ increases to $\infty$, which leads to

$$\int_0^\infty e^{2\pi R x} F(x)\, dx \;\le\; \frac{1}{2\pi R}\exp(2\pi C(K) R^2 \log^{-1/2} R) + O_K(1),$$

which, together with (9) and after taking logarithms, concludes the proof of (8) and the lemma. $\Box$

Our main result can now be proved easily.

*Proof of Theorem* 1: Let $u \in \mathbf{R}^2$ have unit length. The restriction of $g \in C(\mathbf{R}^2)$ along $u$ is defined to be the function (on the real line)

$$(R_u g)(t) \;=\; g(tu).$$

For any $f \in L^1(\mathbf{R}^2)$ we get by an application of Fubini's theorem

$$(R_u \widehat{f})(t) \;=\; (\Pi_u f)^\wedge(t), \quad \text{for all } t \in \mathbf{R}. \tag{10}$$

Assume now that $S$ has the STEINHAUS property. By Lemma 1 we obtain that $\widehat{\mathbf{1}_S}$ vanishes on all circles centered at the origin that contain a lattice point and by a well known result [2, p. 19] the number of those circles with radius $\le R$ (which is the same as the number of integers $\le R^2$ representable as a sum of two squares) is

$$\sim cR^2 \log^{-1/2} R, \quad \text{for a constant } c > 0.$$

If we define the *continuous* function

$$f \;=\; (\Pi_u \mathbf{1}_S) \star \varphi,$$

where $\varphi$ is, say, the indicator function of the unit interval, then the number of zeros of

$$\widehat{f} \;=\; (\Pi_u \mathbf{1}_S)^\wedge \cdot \widehat{\varphi} \;=\; (R_u \widehat{\mathbf{1}_S}) \cdot \widehat{\varphi}$$

in the interval $[-R, R]$ is also at least $\sim cR^2 \log^{-1/2} R$. Define the function

$$F(x) \;=\; \int_x^\infty [(\Pi_u f)(t) + (\Pi_u f)(-t)]\, dt,$$

and notice that it too satisfies (3) with slightly smaller $K$. We can now apply Lemma 3 to the function $f$ and obtain that the number of zeros of $\widehat{f}$ in $[-R, R]$ is at most $C(K) R^2 \log^{-1/2} R$. Choose $K$ large enough to obtain a contradiction. $\Box$

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