

TILINGS BY TRANSLATION

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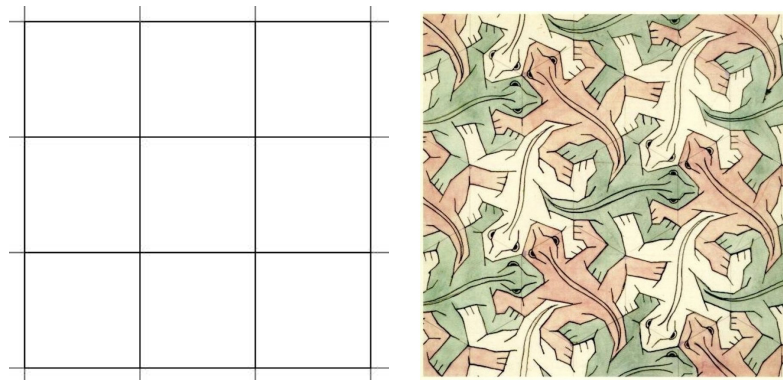
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1. INTRODUCTION

To tile means to cover a given part of space, without overlaps, using a small number, often just one, of different types of objects. When we are covering the floor of a room using identical rectangular tiles (in the ordinary sense of the word) we put down the copies of the tiles in a regular way next to each other, leaving no gaps.

Your floor can be as boring as the rectangular floor on the left or as interesting as the Escher lizard tiling on the right:

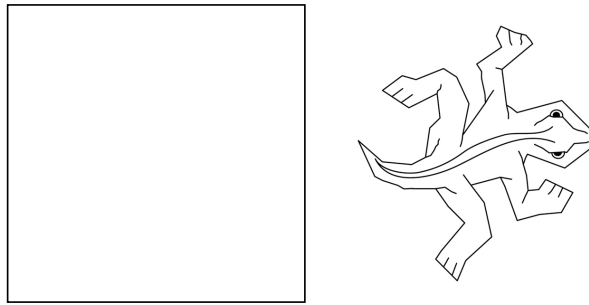


In both these floor tilings there is only one tile used, a square tile on the left and a lizard shape tile

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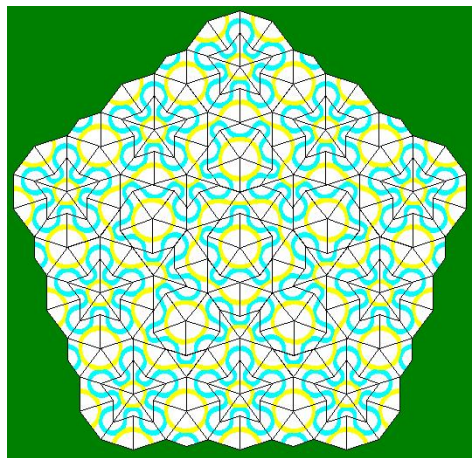
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on the right.

Beyond aesthetics and decoration of the tile there is another difference between these two examples which is very significant from our point of view. To fill the floor using the square tile we only need to translate the tile around; we never have to turn it. This is not the case with the lizard tile. To fill the floor the various copies of the lizard tiles have to be turned appropriately so as to fit perfectly with each other.

It is with the seemingly boring case that we shall occupy ourselves in this survey: the *single* tile that we have at our disposal is *only allowed to translate* in space. We cannot rotate it or reflect it. True, in the vast majority of examples in the literature (see e.g. [13]) that exhibit interesting behavior (such as undecidability or aperiodicity, on which we'll have more to say later) the allowed *group of motions* is usually the full group of rigid motions in space and therefore rotations are allowed (and some times even reflections as well). One would indeed be hard pressed to find eye-catching examples such as the Penrose “kite and dart” aperiodic set of tiles shown below.

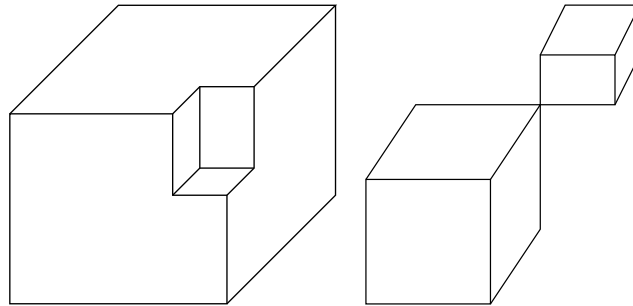


But we have good reasons for restricting ourselves to *tiling by translation* which we hope to expose in this survey. To begin with let us state that tiling by translation is still full of unresolved questions with connections to number theory, Fourier analysis and the theory of computation, and that happens even in dimension 1. If it is hard to imagine how one can seriously study questions of tiling in dimension 1 let us point out that the tiles need not have a “nice” shape. For instance, one can easily show by trial and error that the set $E = \{0, 2, 3, 5\}$ (shown below in red)



can tile the integers \mathbb{Z} by translation only and that such a tiling has period 12. This trial and error process can become slower and slower if one considers larger finite sets of integers and simply asks if they could be used to tile \mathbb{Z} . In fact, it is not known how to do this in any way that is essentially faster than trial and error. This reflects how little we understand this tiling phenomenon.

Here is another example, of a more geometric flavor, to convince the reader that taking away the freedom to rotate and reflect the tile does not take away the fun. For this consider the so called *notched cube* (left)



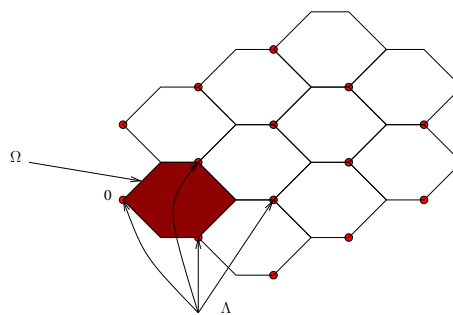
and try to visualize a way to fill 3 dimensional space with it (here it would not help to rotate or reflect it). We are making no assumptions here about the lengths of the rectangular cut (notch) at one corner of the cube or rectangle. Whatever these lengths are this notched cube can indeed tile space by translation [18]. Yet the simplest way we know how to prove this is using Fourier analysis, making absolutely no use of geometric intuition. Same holds for the *extended cube* (above right) and this seems to be even harder to visualize. At least the notched cube “forces” a corner of a copy of the tile to fill the notch in another copy and this can at least get you started but no such restriction is obvious for the extended cube.

By now we should fix the rules of the game. What does it mean for a subset Ω of \mathbb{R}^d to tile \mathbb{R}^d when translated at the locations $\Lambda \subseteq \mathbb{R}^d$ (another *discrete set*)? Simply that the copies

$$\Omega + \lambda, \quad \lambda \in \Lambda,$$

are mutually *non-overlapping*, and *cover the whole space*. Here, as is customary, we denote by $\Omega + \lambda$ the set

$$\{\omega + \lambda : \omega \in \Omega\},$$



or, in other words, the translate of Ω by the vector λ . And what does it mean for two sets to be non-overlapping? One could, for instance, demand that the interiors, but not the boundaries, of the two sets are disjoint. But it turns out that the most convenient way is to demand that the intersection of these two sets has zero volume (or area in dimension $d = 2$, or length in dimension $d = 1$) or, more precisely, zero Lebesgue measure.

2. USING THE FOURIER TRANSFORM

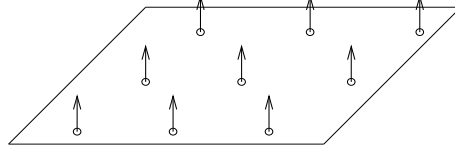
2.1. Tiling in the Fourier domain. Having specified what we mean we turn now to exploiting the technology that we know, and in this case we find it very fruitful to define tiling simply by the equation

$$(1) \quad \sum_{\lambda \in \Lambda} \chi_{\Omega}(x - \lambda) = 1, \quad (\text{almost everywhere for } x \in \mathbb{R}^d).$$

We now have an equation! as a physicist would exclaim. A mathematician should never underestimate the power of formal manipulation and the phenomena it could reveal. So, let us rewrite our equation (ignoring from now on the exception of 0 measure):

$$(2) \quad \chi_\Omega * \delta_\Lambda = 1.$$

Here the $*$ operator denotes convolution and δ_Λ is the *measure*



$$\delta_\Lambda = \sum_{\lambda \in \Lambda} \delta_\lambda,$$

where δ_λ is a unit point mass sitting at point $\lambda \in \mathbb{R}^d$. (We recall that $f * \delta_\lambda(x) = f(x - \lambda)$, so that convolving a function f with δ_λ merely translates the function by the vector λ .) The object δ_Λ is a compact way to encode all information that's contained in the set of translates Λ in a way that we can operate algebraically on it.

An equation such as (2) of course begs to be subjected to the Fourier Transform¹ as the Fourier Transform (FT) behaves so nicely with convolution (the FT of a convolution is the pointwise product of the FTs of the convolution factors: $\widehat{f * g} = \widehat{f} \widehat{g}$). The definition of the FT \widehat{f} of a function f that we use is

$$(3) \quad \widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i \xi \cdot x} dx, \quad (\xi \in \mathbb{R}^d, \int_{\mathbb{R}^d} |f| < \infty).$$

This definition is sufficient for the function $f = \chi_\Omega$ which is integrable but one needs to define the FT of a measure by duality (see for instance [22] for the details). Ignoring such subtleties we take the FT of both sides of (2) to get

$$(4) \quad \widehat{\chi_\Omega} \cdot \widehat{\delta_\Lambda} = \delta_0.$$

The measure δ_0 on the right is just a point mass at point 0. The support (where it is “nonzero”) of this is $\{0\}$ and therefore that must be the support of the left hand side of (4). Since that is a product it follows that, apart from point 0, wherever $\widehat{\chi_\Omega}$ is nonzero, $\widehat{\delta_\Lambda}$ must be zero so as to kill the product. Let us summarize this in the following necessary (and often sufficient) condition for tiling:

$$(5) \quad \text{supp } \widehat{\delta_\Lambda} \subseteq \{\widehat{\chi_\Omega} = 0\} \cup \{0\}.$$

A formal proof of (5) can be looked up in [22]. Let us make no comment here about when this condition is sufficient. But we must point out that the object $\widehat{\delta_\Lambda}$ is not necessarily a measure but a so called *tempered distribution* [35].

2.2. A problem from the Scottish Book. Nice reformulation but what could this buy for us?

Following the time-honored tradition of our trade, let us first generalize: whatever we've said so far holds just as well for an (almost) arbitrary function f (nonnegative, integrable would do) in place of χ_Ω . In other words, if one has

$$(6) \quad \sum_{\lambda \in \Lambda} f(x - \lambda) = \ell = \text{Const.},$$

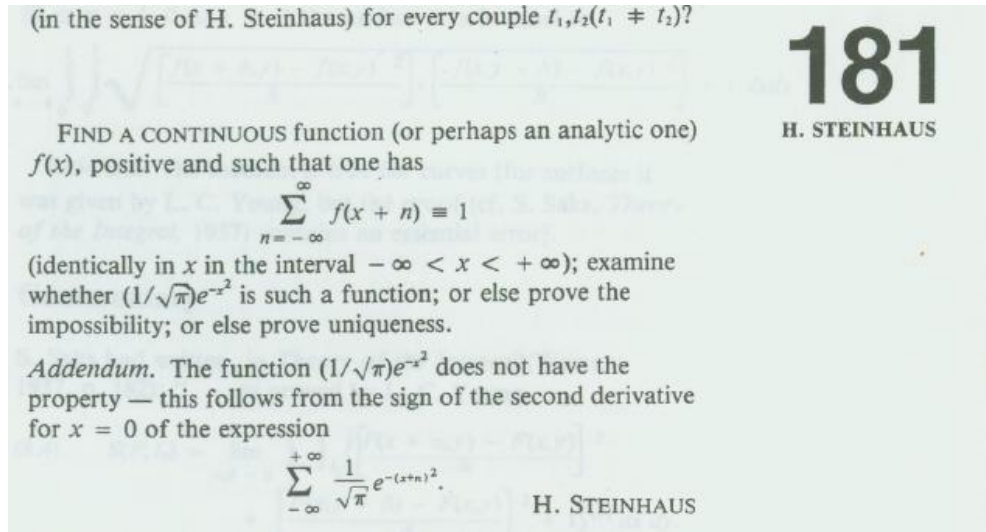
for almost every $x \in \mathbb{R}^d$ then it follows that

$$(7) \quad \text{supp } \widehat{\delta_\Lambda} \subseteq \mathcal{Z}(\widehat{f}) \cup \{0\} \quad \text{where } \mathcal{Z}(\widehat{f}) = \{\widehat{f} = 0\}.$$

¹This of course depends on what training *you* have been subjected to.

Whenever (6) holds we will say from now on that f tiles space with Λ at level ℓ .

Let us apply our new tool to Problem No 181 in the famous Scottish Book² [31], a problem posed by H. Steinhaus and partly solved by him around 1939.



In our language the question essentially was if there is a strictly positive, continuous and, if possible, analytic function f that tiles \mathbb{R} with \mathbb{Z} .

Whoever has been exposed even a little to the FT knows that e^{-x^2} has the nice property that its FT transform is essentially itself (times a constant and possibly rescaled, depending on the normalizations used in the definition of the FT) and therefore it has no zeros at all. It follows that (7) has no chance of holding as $\widehat{\delta}_{\Lambda}$ has no place to be supported at apart from 0, and it's not a constant. And this argument does not use the fact that $\Lambda = \mathbb{Z}$ so e^{-x^2} does not tile \mathbb{R} with *any* set of translates.

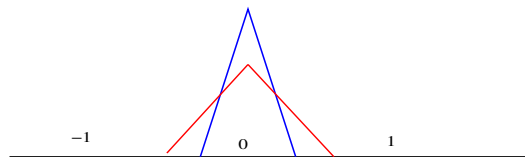
An analytic solution to this question of Steinhaus can also be provided using our technology. First of all, let us make the important remark that

$$(8) \quad \widehat{\delta}_{\mathbb{Z}^d} = \delta_{\mathbb{Z}^d}.$$

This is the deservedly famous Poisson Summation Formula [36, Ch. 9]

$$\sum_n f(n) = \sum_n \widehat{f}(n)$$

in disguise. In this case, when $\widehat{\delta}_{\Lambda}$ is a measure, condition (7) is also sufficient for tiling [22], so it is sufficient to find a positive function f whose FT vanishes on $\mathbb{Z} \setminus \{0\}$. Analyticity of f will be guaranteed if \widehat{f} has compact support. It is then easy to check that one can take as \widehat{f} the sum of two triangle functions with incommensurable base lengths,

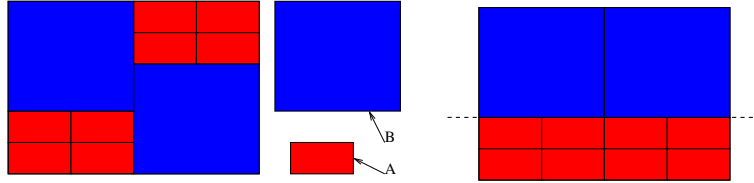


as the FT of each such triangle is a nonnegative function (a Fejér kernel) whose zeros are at integer multiples of the reciprocal of its half-base length. Since these will never match for the two Fejér kernels their sum is always positive. Since we also take the bases of the two triangles to be supported in $(-1, 1)$ it follows that (7) holds and we have tiling.

²A collection of problems from the Scottish Café in Lwów (then in Poland, now Lviv in the Ukraine) where a number of important mathematicians met for years exchanging problems. These problems, and whatever solutions, were recorded in a notebook that formed the basis for [31].

2.3. Filling a box with two types of bricks. Let us give another amusing application of the FT method to tiling problems. This will be a Fourier analytic proof [21] of a result of Bower and Michael [5]. Suppose that you have two types of rectangular bricks at your disposal, type *A* of dimensions $a_1 \times a_2$ and type *B* of dimensions $b_1 \times b_2$ (we're stating everything in dimension 2 but everything works in any dimension) and your task is to tile a rectangular box, say $Q = (-1/2, 1/2)^2$, with copies of bricks A and B. The bricks may be translated but not rotated.

We will show that this is possible if and only if you can cut Q along the x or along the y direction into two rectangles each of which can be tiled using bricks of one type only. A generic tiling of the type shown below left implies the existence of a *separated* tiling as shown below right.

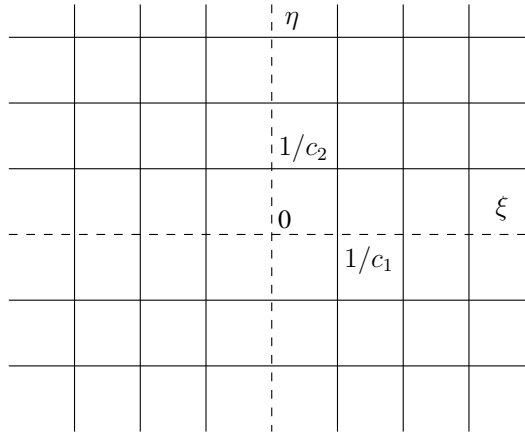


In other words, having two types of bricks at your disposal does not demand much ingenuity on your side in order to exploit for tiling purposes. If you cannot do it in an obvious way (i.e. without mixing the two types of bricks) then it cannot be done at all.

A simple calculation shows that if $C = (-\frac{c_1}{2}, \frac{c_1}{2}) \times (-\frac{c_2}{2}, \frac{c_2}{2})$ is a centered $c_1 \times c_2$ box then

$$\widehat{\chi}_C(\xi, \eta) = \frac{\sin(\pi c_1 \xi)}{\xi} \cdot \frac{\sin(\pi c_2 \eta)}{\eta},$$

and therefore $\widehat{\chi}_C$ vanishes at those points, shown below as solid lines,



where the ξ or the η coordinate is a non-zero multiple of $1/c_1$ or $1/c_2$ respectively

Suppose now that we can tile Q by translating copies of brick *A* to the locations T and copies of brick *B* to the locations S . In other words

$$(9) \quad \chi_Q(x) = \sum_{t \in T} \chi_A(x - t) + \sum_{s \in S} \chi_B(x - s).$$

which we rewrite as

$$\chi_Q = \chi_A * \delta_T + \chi_B * \delta_S$$

and take the FT of both sides to get

$$(10) \quad \widehat{\chi}_Q(\xi, \eta) = \phi_T(\xi, \eta) \widehat{\chi}_A(\xi, \eta) + \phi_S(\xi, \eta) \widehat{\chi}_B(\xi, \eta),$$

where $\phi_T = \widehat{\delta}_T$ and $\phi_S = \widehat{\delta}_S$ are two *trigonometric polynomials*. Since $Q = (-1/2, 1/2)^2$ we have

$$(11) \quad \mathcal{Z}(\widehat{\chi}_Q) = \{\widehat{\chi}_Q = 0\} = \{(\xi, \eta) : \xi \in \mathbb{Z} \setminus \{0\} \text{ or } \eta \in \mathbb{Z} \setminus \{0\}\}.$$

But, because of (10), $\widehat{\chi}_Q$ must vanish on the common zeros of $\widehat{\chi}_A$ and $\widehat{\chi}_B$, for instance at the points $(1/a_1, 1/b_2)$ and $(1/b_1, 1/a_2)$, which implies, because of (11)

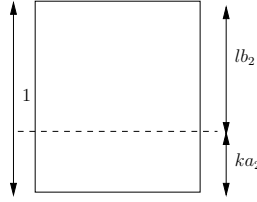
$$(12) \quad (1/a_1 \in \mathbb{Z} \text{ or } 1/b_2 \in \mathbb{Z}) \text{ and } (1/b_1 \in \mathbb{Z} \text{ or } 1/a_2 \in \mathbb{Z}).$$

If (12) is true because $1/a_1, 1/a_2 \in \mathbb{Z}$ then brick A alone can fill Q . Similarly if (12) is satisfied with $1/b_1, 1/b_2 \in \mathbb{Z}$ then brick B alone suffices.

What happens if $1/a_1, 1/b_1 \in \mathbb{Z}$? Since we have assumed a tiling of Q in (9) it follows, by traversing the y -axis, that there are nonnegative integers k, l such that

$$1 = ka_2 + lb_2.$$

Cut now the Q box parallel to the x -axis at height ka_2 from the bottom as show here:



Now it is clear that brick A can fill the lower box (since $1/a_1 \in \mathbb{Z}$) and brick B can fill the upper box (since $1/b_1 \in \mathbb{Z}$). The remaining case $1/a_2, 1/b_2 \in \mathbb{Z}$ is treated similarly.

3. DISCRETE TILINGS

3.1. Tilings of the integers and periodicity. Let us now focus on tiling the integers. Let $A \subseteq \mathbb{Z}$ be a finite set and $\Lambda \in \mathbb{Z}$. We say that A tiles \mathbb{Z} with Λ at level ℓ if the copies $A + \lambda$, $\lambda \in \Lambda$, cover every integer exactly ℓ times. In other words

$$\sum_{\lambda \in \Lambda} \chi_A(x - \lambda) = \ell, \text{ for all } x \in \mathbb{Z}.$$

We denote this situation as $A + \Lambda = \ell\mathbb{Z}$.

We say that a tiling is periodic with period $t \in \mathbb{Z}$ if $\Lambda + t = \Lambda$. It is a basic fact proved by Newman [32] that all tilings of the integers at level 1 are periodic.

Indeed, suppose that $A = \{0 = a_1 < \dots < a_k\}$ is a finite set (we may freely translate A without changing the tiling or the periodicity property so we assume it starts at 0) and $A + \Lambda = \mathbb{Z}$ is a tiling at level 1. Fix any $x \in \mathbb{Z}$ and write $W_x = x, x + 1, \dots, x + a_k - 1$ for the “window” of width $a_k - 1$ (one less than A) starting at x . We claim that the set $\Lambda \cap W_x$ determines Λ . Let us show that it determines Λ to the right of $x + a_k - 1$. It will determine Λ to the left of x by the same argument.

It is enough to decide, looking at $\Lambda \cap W_x$ only, if $x + a_k \in \Lambda$ or not. (We then repeat for $x + a_k + 1$ and so on.) Observe that for any $\lambda \in \Lambda \cap (-\infty, x)$ the set $A + \lambda$ is contained in $(-\infty, x + a_k - 1]$, so any such copy cannot be used to cover $x + a_k$. Clearly no copy of the form $A + \lambda$ for $\lambda > x + a_k$ can be used for that purpose too. We conclude that $x + a_k$ is covered by some copy $A + \lambda$ with $\lambda \in W_x \cup \{x + a_k\}$. Inspecting $\Lambda \cap W_x$ we can tell if the relevant λ is in W_x or not. If it is in W_x then $x + a_k \notin \Lambda$ since this would lead to the copies $A + \lambda$ and $A + (x + a_k)$ to overlap at $x + a_k$. If it is not in W_x then necessarily $x + a_k \in \Lambda$, and this concludes the proof of the claim.

How many different values can the set $\Lambda \cap W_x$ take? Clearly it can take at most 2^{a_k} different values as there are two choices (in Λ or not in Λ) for each $x \in W_x$. This means that there are two different $x, y \in \{0, 1, \dots, 2^{a_k}\}$ for which $\Lambda \cap W_x$ is a translate of $\Lambda \cap W_y$. It follows that $\Lambda + (y - x) = \Lambda$. We have proved that every tiling has a period which is at most 2^D where D is the diameter of the tile.

There is a similar result for tilings of the continuous line by translates of a function [23, 28, 22]. Combinatorial arguments do not seem to be enough here and the Fourier

analytic technology along with some deep results of Harmonic Analysis are being used for the proof.

3.2. Tilings of the finite cyclic group. The fact that a tiling of the integers is periodic allows us to view it as a tiling on a smaller structure, a cyclic group. Indeed, assume that the tiling $A + \Lambda = \mathbb{Z}$ has period n , that is $\Lambda + n = \Lambda$. Define the set

$$\tilde{\Lambda} = \Lambda \bmod n \subseteq \{0, 1, \dots, n-1\}$$

by taking for each $\lambda \in \Lambda$ its residue mod n . It follows from the n -periodicity of Λ that $\Lambda = \tilde{\Lambda} + n\mathbb{Z}$ (this is again a tiling or a *direct sum*: every element of Λ can be written in a unique way as an element of $\tilde{\Lambda}$ plus an element of $n\mathbb{Z}$). Hence we have

$$\mathbb{Z} = A + \Lambda = A + \tilde{\Lambda} + n\mathbb{Z}$$

with all sums being direct. Taking quotients we obtain that the cyclic group $\mathbb{Z}_n = \mathbb{Z}/(n\mathbb{Z})$ can be written as a direct sum (tiling)

$$(13) \quad \mathbb{Z}_n = A + \tilde{\Lambda}.$$

In this case we obviously have $n = |A| \cdot |\tilde{\Lambda}|$.

Let us stop here to make two side remarks about periodicity. The first is that tilings of the cyclic group \mathbb{Z}_n can also be periodic. Indeed, assume $\mathbb{Z}_n = A + \tilde{\Lambda}$ and there exists $k \in \mathbb{Z}_n \setminus \{0\}$ such that A is periodic with period k , i.e. $A + k = A$. Then we can reduce the set A modulo k , $\tilde{A} = A \bmod k \subseteq \{0, 1, \dots, k-1\}$, and conclude as above that $\mathbb{Z}_k = \tilde{A} + \tilde{\Lambda}$. Therefore, periodic tilings of \mathbb{Z}_n can be regarded as tilings of a smaller group \mathbb{Z}_k (it is trivial to see that k automatically divides n). It is thus natural to ask whether certain cyclic groups \mathbb{Z}_n admit *only* periodic tilings, i.e. whenever $\mathbb{Z}_n = A + \tilde{\Lambda}$, then either A or $\tilde{\Lambda}$ *must* be periodic. These were called “good” groups by Hajós [14] (the notion also makes sense in the more general setting of finite Abelian groups, not only cyclic groups). It turns out that some groups indeed have this property, and Sands completed the classification of good groups in [38, 39]. In particular, the good groups that are cyclic are \mathbb{Z}_n where n divides one of $pqrs$, p^2qr , p^2q^2 or p^mq , where p, q, r, s are any distinct primes. The cyclic group of the smallest order which is not good is \mathbb{Z}_{72} .

The other remark concerns the connection of tilings of cyclic groups to *music composition*. As explained above, periodic tilings of \mathbb{Z}_n are mathematically speaking less interesting because they can be considered as tilings of some smaller group \mathbb{Z}_k . It turns out that non-periodic tilings of a cyclic group \mathbb{Z}_n are also more interesting from an *aesthetic* point of view, and they are called *Vuza-cansons* in the musical community. The interaction of mathematical theory and musical background has been rather intense in recent years [3, 1, 2, 10, 42]. Finding all non-periodic tilings of \mathbb{Z}_n is therefore motivated by contemporary music compositions. Of course, the problem makes sense only if \mathbb{Z}_n is not a good group (otherwise all tilings are periodic). Friperntinger [9] achieved this task for $n = 72, 108$ while the authors [26] gave an efficient algorithm to settle the case $n = 144$ (the algorithm is likely to work for other values like $n = 120, 180, 200, 216$, beyond which the task simply seems hopeless).

Let us now return to the analysis of tilings of the cyclic group \mathbb{Z}_n . The Fourier condition (5) for tiling takes exactly the same form here and is much simpler to prove as no subtle analysis is required (no integrals, only finite sums are involved). The Fourier transform of a function $f : \mathbb{Z}_n \rightarrow \mathbb{C}$ is defined as the function $\hat{f} : \mathbb{Z}_n \rightarrow \mathbb{C}$ given by

$$\hat{f}(k) = \sum_{j=0}^{n-1} f(j) \zeta_n^{-kj},$$

where $\zeta_n = e^{2\pi i/n}$ is a primitive n -th root of unity.

In the finite case the roles of the tile A and the set of translations $\tilde{\Lambda}$ in (13) can be interchanged, so let us adopt a more symmetric notation

$$\mathbb{Z}_n = A + B,$$

in which $A, B \subseteq \mathbb{Z}_n$ are just two subsets, necessarily satisfying $|A| \cdot |B| = n$, and such that every element of \mathbb{Z}_n can be written uniquely as a sum of an element of A and an element of B . The Fourier condition now takes the form

$$(14) \quad \mathbb{Z}_n = \{0\} \cup \mathcal{Z}(\widehat{\chi}_A) \cup \mathcal{Z}(\widehat{\chi}_B).$$

As we saw in the section §3.1 every tiling of \mathbb{Z} has a period which is at most 2^D , where D is the diameter of the tile. We will see in this section how the easy combinatorial argument of §3.1 can be replaced with an argument that has the Fourier condition (14) as a starting point, uses some well-known number-theoretic facts (properties of cyclotomic polynomials) and gives much better results. We will describe the result in [20] and by I. Ruzsa in an appendix in [41] since it is simpler than the current best results in [4].

So, suppose that $A \subseteq \{0, 1, \dots, D\}$ is a set of integers of diameter $\leq D$ and that A tiles \mathbb{Z} with period M . This implies that A (to be precise, A reduced mod M) tiles the cyclic group \mathbb{Z}_M

$$\mathbb{Z}_M = A + B.$$

Assume also that M is the least period. This implies that if $g \in \mathbb{Z}_M$ is such that $B = B + g$ then $g = 0$. (Otherwise the original tiling would be periodic with period g .)

Now we use the Fourier transform without really mentioning it, using instead polynomial terminology. Let

$$A(x) = \sum_{a \in A} x^a, \quad B(x) = \sum_{b \in B} x^b,$$

be the polynomials defined by A and B . It is very easy to see that the tiling condition $\mathbb{Z}_M = A + B$ is the same as

$$A(x)B(x) = 1 + x + x^2 + \dots + x^{M-1} \pmod{x^M - 1},$$

or, in other words, that

$$(15) \quad x^M - 1 \text{ divides } A(x)B(x) - \frac{x^M - 1}{x - 1},$$

and this implies that all M -th roots of unity except 1 are roots of the product $A(x)B(x)$. (This last statement is the equivalent of (14) in algebraic language.)

The M -th roots of unity $\zeta_M^j = e^{2\pi i j/M}$, $j = 0, 1, \dots, M-1$, are grouped into *cyclotomic classes*. Two roots ζ_M^j, ζ_M^k belong to the same cyclotomic class if and only if the greatest common divisors (j, M) and (k, M) are the same. If a root of unity in a certain cyclotomic class is a root of a polynomial with integer or rational coefficients then so are all the other roots in the same cyclotomic class. If d is a positive integer then the d -th cyclotomic polynomial is the monic polynomial which has as roots all the *primitive* d -th roots of unity

$$\Phi_d(x) = \prod_{1 \leq j < d, (j,d)=1} (x - e^{2\pi i j/d}).$$

We have $\deg \Phi_d(x) = \phi(d)$ (the Euler function) and we will use the estimate [15, p. 267]

$$(16) \quad C \frac{d}{\log \log d} \leq \phi(d) \leq d,$$

which holds for some constant C and all large enough d . It turns out that each Φ_d is an irreducible polynomial with integer coefficients and one can write

$$x^M - 1 = \prod_{d \text{ divides } M} \Phi_d(x).$$

Coupled with (15) this implies that if $d > 1$ is a divisor of M then $\Phi_d(x)$ divides the product $A(x)B(x)$.

Let now $\Phi_{s_1}(x), \dots, \Phi_{s_k}(x)$ be all cyclotomic polynomials $\Phi_s(x)$ with $s > 1$ that divide $A(x)$, written *once* each and numbered so that $1 < s_1 < s_2 < \dots < s_k$. Since $\deg \Phi_s = \phi(s)$ it follows that

$$(17) \quad \phi(s_1) + \dots + \phi(s_k) \leq \deg A(x) \leq D.$$

We have

$$\begin{aligned} \sum_{i=1}^k s_i &\leq C \sum_{i=1}^k \phi(s_i) \log \log s_i \text{ (from (16))} \\ &\leq C \sum_{i=1}^k \phi(s_i) \log \log D \\ &\leq CD \log \log D \quad \text{(from (17)).} \end{aligned}$$

From this inequality and using the fact that the s_j are *different* integers it follows that

$$(18) \quad k \leq C\sqrt{D \log \log D},$$

as obviously the worst case is if the s_j are k consecutive integers and in that case their sum is $\gtrsim k^2$.

Define now $t = \prod_{i=1}^k s_i$, so that all cyclotomic polynomials that divide $A(x)$ are also divisors of $x^t - 1$. It follows that

$$x^M - 1 \text{ divides } (x^t - 1)B(x)$$

and this means precisely that B has period t , hence $t \geq M$ as we assumed M to be minimal. Using (18) and the bound $s_j = O(D^2)$, for instance, we get the following bound for the least period M :

$$(19) \quad M \leq t \leq \exp(C\sqrt{D} \log D \sqrt{\log \log D}),$$

for an appropriate constant C . This is a much better bound, in terms of its dependence on D , that the bound $M \leq 2^D$ that we obtained with combinatorics alone.

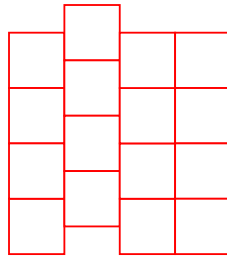
3.3. Periodicity in two dimensions and computability. In contrast to the one-dimensional case, where every tiling by a finite subset of \mathbb{Z} is periodic, it is very easy to see that in \mathbb{Z}^2 there are tilings which are not periodic. One has to be a little careful with periodicity in dimension 2 and higher though. We shall call a set $A \subseteq \mathbb{Z}^d$ periodic if there exists a full lattice of periods, i.e. if there exist d linearly independent vectors u_1, \dots, u_d which are all periods of the set

$$A + u_1 = A + u_2 = \dots = A + u_d = A.$$

If that happens then every integer linear combination of the u_j is also a period or, in other words, the set A has a full-dimensional lattice of periods

$$\Lambda = \text{span}_{\mathbb{Z}} \{u_1, u_2, \dots, u_d\}.$$

Thus in dimension 2 a set may have a period but not two linearly independent periods, in which case we do not call it periodic. One example is the set $\mathbb{Z} \times \{0\} \subseteq \mathbb{Z}^2$. A tiling of \mathbb{Z}^2 by a finite subset which is not periodic is very easy to construct.



One takes A to be a square, for instance the set $\{(0,0), (1,0), (1,0), (1,1)\}$ and perturbs the usual tiling of it by shifting one column only up by one unit. This destroys the period $(2,0)$ of the tiling along the x -axis, but leaves the period $(0,2)$ along the y -axis intact.

It is not entirely obvious how to construct a tiling of \mathbb{Z}^2 which has no period at all, but it can be done and let us briefly describe how. The key idea is to start again with the usual square tiling and do something to it so as to destroy all periods, not just the periods along one axis. The way to do this is to simultaneously shift a horizontal and a vertical column. This is not of course possible with the square tile we used before, as simple experimentation will convince you. But it can be done if we “interleave” four tilings of the type shown above. Take our tile to be the set

$$A = \{(0,0), (2,0), (0,2), (2,2)\}$$

which is what we had before only dilated by a factor of two. Now tile $(2\mathbb{Z})^2$ using A in the usual way, that is by translating A to the locations $(4\mathbb{Z})^2$. Using this tiling one can get a tiling of \mathbb{Z}^2 by tiling the four sets (cosets of the subgroup $(2\mathbb{Z})^2$ in \mathbb{Z}^2)

$$(2\mathbb{Z})^2, (2\mathbb{Z})^2 + (1,0), (2\mathbb{Z})^2 + (0,1), (2\mathbb{Z})^2 + (1,1)$$

in exactly the same way. But the smart way to do it is to use the nice tiling for the first two of the above four cosets, then use, for tiling $(2\mathbb{Z})^2 + (0,1)$, a tiling like that shown in the figure above (which destroys the horizontal periods) and finally use for tiling $(2\mathbb{Z})^2 + (1,1)$ a similar tiling which destroys the vertical periods. The tiling that we constructed in this manner has no period vectors at all.

Having established that in two dimensions there are tilings with no periods at all, let us now remark that it is very different to ask for *tiles* which are aperiodic, i.e. for tiles which can tile but only in a way that admits no periods. In fact, the answer to this question is not known if we insist that only translations are allowed.

Conjecture 3.1 (Lagarias and Wang [28]). *If a finite subset $A \subseteq \mathbb{Z}^2$ can tile by translation then it can also tile in a periodic way.*

Let us point out that if more freedom than translation is allowed then tiles (or sets of tiles) are known which can only tile aperiodically, so the above conjecture, if true, would mean that restricting the allowed motions to translations makes a big difference in this respect.

Let us also remark that it can be proved, at least in dimension 2 [33], that if a set admits a tiling with just one period then it also admits a (fully) periodic tiling.

The property of periodicity is of great importance to the issue of computability, namely to whether a computer can decide if a finite set A admits tiling by translation. In still more words, we are interested to know if there is a computer algorithm (a Turing machine for purists) which, given a finite subset A of \mathbb{Z}^2 , will decide in finite time if there exists a tiling complement of A or not, i.e. if there exists $B \subseteq \mathbb{Z}^2$ such that $A + B = \mathbb{Z}^2$ is a tiling.

We do not require our algorithm to:

- run fast (only to finish at some point)
- find such a tiling complement B if it exists.

Especially for the second point notice that it would not make sense to want to know B as such a set is an infinite object and there is no a priori reason for it to be describable in some finite manner.

It is easy to give an algorithm that would answer NO if A is not a tile but would run forever if A is a tile. Here is how this can be done. For each $n = 1, 2, \dots$, decide by trial and error if there is a finite set B such that $A + B$ covers the (discrete) square $Q_n = [-n, n]^2 \cap \mathbb{Z}^2$ in a non-overlapping way. There is a finite number of such sets B to try as it does not matter what B is far away from Q_n . If such a B exists move on to the next n . If not then declare A a non-tile and stop. The correctness of this algorithm (i.e., the fact that the algorithm will stop for any non-tile A) follows from a fairly simple diagonal argument: if a finite A can cover in a nonoverlapping way every Q_n then it can also tile \mathbb{Z}^2 . We invite the reader to think over why this is so.

The hard thing of course is to have an algorithm that always halts and say YES if A is a tile and NO if it is not a tile. Let us point out here that in various other tiling situations such an algorithm does not exist. For instance [12] there is no algorithm which, given a finite collection of subsets of \mathbb{Z}^2 , decides if these can tile the plane by translation. This remains true even if the number of subsets remains bounded (but large).

The connection with periodicity [34] is that the periodic tiling conjecture above implies decidability. Indeed, let us assume that any set that tiles can also tile periodically. (We emphasize here that we are making no assumption about the size of the periods or their dependence on the tile.) This means that it is equivalent to decide if a given set A admits periodic tilings.

Let us assume that $A \subseteq [0, D]^2$. For $n = 1, 2, \dots$, we can clearly find all subsets B of $[-n - 2D, n + 2D]^2$ such that $A + B$ is a nonoverlapping covering of $Q_n = [-n, n]^2$. We can do this, slowly but surely, by just trying all eligible subsets. If we find none then clearly our set is not a tile, we say NO, and we stop. If we do find some we look among them to find a set B which “can be extended periodically”. How does such a set look like?

If an infinite set $\tilde{B} \subseteq \mathbb{Z}^2$ is periodic with linearly independent period vectors $u_1, u_2 \in \mathbb{Z}^2$ then we can always find two vectors

$$\tilde{u}_1 = (a, 0), \quad \tilde{u}_2 = (0, b), \quad a, b \in \{1, 2, 3, \dots\},$$

which are also periods (in other words, every lattice in \mathbb{Z}^2 contains a sublattice generated by two vectors on the x - and y -axes). The mapping $(x, y) \rightarrow (x \bmod a, y \bmod b)$ maps \tilde{B} to a set

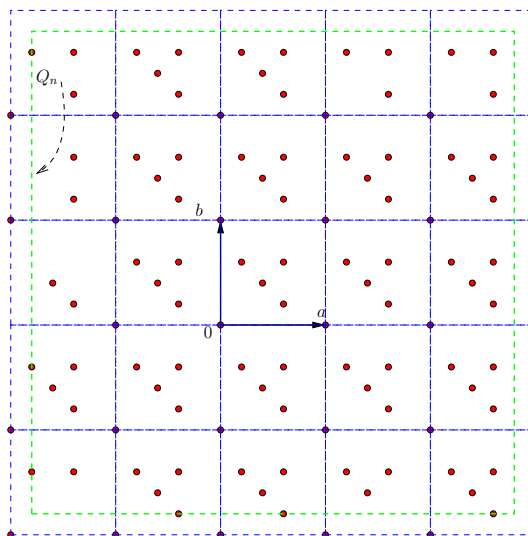
$$B' \subseteq \{0, 1, \dots, a - 1\} \times \{0, 1, \dots, b - 1\}$$

and the periodicity of \tilde{B} means exactly that

$$\tilde{B} = (a\mathbb{Z}) \times (b\mathbb{Z}) + B'$$

is a direct sum, i.e. every element of \tilde{B} can be written uniquely in the form $(am, bn) + b'$, where $m, n \in \mathbb{Z}, b' \in B'$.

Suppose now that our algorithm looks at a finite set B which is a finite part of a periodic set \tilde{B} . This finite part of \tilde{B} arises if we keep those elements of \tilde{B} which are used in a non-overlapping covering of Q_n (shown as a dashed green rectangle). If n is large enough then the set B will look something like this:



That is, the set B will contain several “full periods” (the dashed blue $a \times b$ rectangles) \tilde{B} plus a few incomplete periods near the border of Q_n . Suppose now that we have a $k \times k$ block of full periods in the set B and that k is large enough that both ka and kb are much larger than the diameter of A , say larger than $10D$. Then we can just extend this block periodically (with periods $(a, 0)$ and $(0, b)$) and we will get a periodic tiling complement of A . The reason is that (a) no overlaps arise in this manner or they would have shown up in the $k \times k$ block already and that (b) we have covering of everything as clearly the $k \times k$ block suffices to cover the square of side $5D$ with the same center.

Thus our algorithm will, sooner or later, either discover that A cannot tile (since it will not be able to tile a finite region) or it will discover that A can tile a large region with a tiling complement that looks like the one in the figure above, and which can therefore be extended periodically to a full tiling of \mathbb{Z}^2 . This means that our algorithm will always stop and give us a correct YES or a correct NO. We have not given any bound on the running time though.

4. TILES, SPECTRAL SETS AND COMPLEX HADAMARD MATRICES

In this last section we introduce the notion of spectral sets and describe how it is related to tilings and complex Hadamard matrices. For the sake of simplicity, we will restrict our attention to finite groups, in particular to cyclic groups \mathbb{Z}_n and its powers \mathbb{Z}_n^d .

Although we have seen in the sections above that many fundamental problems about translational tilings remain open, the reader will agree with us that the notion of tiling is very intuitive and easy to grasp. To the contrary, the definition of spectral sets is somewhat more abstract. In order to make it more down-to-earth, we will not use the standard definition here, but first introduce complex Hadamard matrices and then use them to define spectral sets.

What are Hadamard matrices? Classically, they are square matrices consisting of elements ± 1 only, such that the rows (and thus also the columns) are orthogonal to each other. Complex Hadamard matrices are a natural generalization of this concept. A $k \times k$ matrix H is a complex Hadamard matrix if its entries are complex numbers of modulus 1, and the rows (and thus the columns) are orthogonal. Recall here that orthogonality is understood with respect to the *complex* scalar product, i.e. you need to conjugate in one of the components, the scalar product of (z_1, \dots, z_k) and (u_1, \dots, u_k) being $\sum_{j=1}^k \bar{z}_j u_j$.

It is well-known that a $k \times k$ real Hadamard matrix can only exist if k is divisible by 4. On the contrary, complex Hadamard matrices exist in all dimension. Indeed the *Fourier*

matrix F_k defined as

$$F_k := \begin{bmatrix} 1 & 1 & \cdot & \cdot & \cdot & 1 \\ 1 & \zeta_k & \zeta_k^2 & \cdot & \cdot & \zeta_k^{k-1} \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \zeta_k^{jm} & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \zeta_k^{k-1} & \cdot & \cdot & \cdot & \zeta_k \end{bmatrix}$$

where $\zeta_k = e^{2\pi i/k}$, is a complex Hadamard matrix for every k .

Each element $h_{j,m}$ of a $k \times k$ complex Hadamard matrix H is of the form $e^{2i\pi\rho_{j,m}}$ where $\rho_{j,m} \in [0, 1)$. We will call the real $k \times k$ matrix R formed by the angles $\rho_{j,m}$ the (entry-wise) *logarithm* of H . In notation, $R = \frac{1}{2i\pi}\text{LOG}(H)$ or $H = \text{EXP}(2i\pi R)$ (note that both these operations are meant entry-wise). For a real matrix R such that $\text{EXP}(2i\pi R)$ is complex Hadamard, we will use the terminology that R is a *log-Hadamard* matrix.

Complex Hadamard matrices play an important role in quantum information theory, in particular in the construction of teleportation and dense coding schemes [43], among other things. An online catalogue of all known families of complex Hadamard matrices is available at [6].

Let us now turn to the definition of spectral sets. We identify elements of \mathbb{Z}_n^d with column vectors, each coordinate being in the range $\{0, 1, \dots, n-1\}$. Thus, a set $S \subset \mathbb{Z}_n^d$ with r elements can be identified with an $d \times r$ matrix, the columns of which are the elements of S (the order of elements does not matter). We will abuse notation and denote this matrix also by S . We also identify $\widehat{\mathbb{Z}_n^d}$ with row vectors whose coordinates are in $\{0, 1, \dots, n-1\}$. Accordingly, we will use the notation $\widehat{\mathbb{Z}_n^d} = \mathbb{Z}_n^{d\top}$ (the \top meaning transposition). Two sets $S \subset \mathbb{Z}_n^d$ and $Q \subset \mathbb{Z}_n^{d\top}$, both of them having r elements, are called a *spectral pair* if the matrix $\frac{1}{n}QS$ is an $r \times r$ log-Hadamard matrix. In this case Q is called the *spectrum* of S , and S is called a *spectral set*. We remark that $S^\top Q^\top = (QS)^\top$ (where S^\top and Q^\top denote the transposed matrices) is then automatically also a log-Hadamard matrix, so we obtain that S is a spectrum of Q , which justifies the symmetry of the terminology "spectral pair".

For example, the whole group $\mathbb{Z}_n = \{0, 1, \dots, n-1\}$ is spectral, and its spectrum is the whole dual group \mathbb{Z}_n^\top , giving rise to the logarithm of the Fourier matrix, $\frac{1}{2i\pi}\text{LOG}(F_n) = \frac{1}{n}\mathbb{Z}_n^\top\mathbb{Z}_n$. As a less trivial example, take the set $S = \{0, 1, 2\}$ in \mathbb{Z}_6 . Then $Q = \{0, 2, 4\}^\top$ is a spectrum of S , and $\frac{1}{6}QS = \frac{1}{2i\pi}\text{LOG}(F_3)$, the logarithm of the 3×3 Fourier matrix.

We saw that the tiling $\mathbb{Z}_n = A + B$ is characterized by $|A| \cdot |B| = n$ and the Fourier condition $\mathbb{Z}_n = \{0\} \cup \mathcal{Z}(\widehat{\chi}_A) \cup \mathcal{Z}(\widehat{\chi}_B)$ in equation (14). After some inspection one sees that spectral sets also admit a characterization by Fourier analysis. Indeed, using the definition above we obtain that $S \subset \mathbb{Z}_n^d$, $|S| = r$, is spectral if and only if there exists $Q \subset \mathbb{Z}_n^{d\top}$, such that $|Q| = r$ and

$$(20) \quad q_j - q_k \in \mathcal{Z}(\widehat{\chi}_S)$$

for each $q_j \neq q_k \in Q$. (This is an equivalent way of saying that the j th and k th rows of the matrix $\text{EXP}(\frac{2i\pi}{n}QS)$ are orthogonal.)

What are the connections of tiles and spectral sets? Notice that by the orthogonality conditions the matrix $U = \frac{1}{\sqrt{r}}\text{EXP}(\frac{2i\pi}{n}QS)$ is unitary. This implies the following important fact (for a formal proof, see [25]):

$$(21) \quad \sum_{q \in Q} |\widehat{\chi}_S|^2(x - q) = |S|^2 = r^2,$$

for all $x \in \mathbb{Z}_n^{d\top}$. In other words, the function $|\widehat{\chi}_S|^2$ tiles $\widehat{\mathbb{Z}_n^d}$ at level $|S|^2$ when translated at the locations Q . One can also say that the set Q tiles the group $\widehat{\mathbb{Z}_n^d}$ with the *weighted*

translates defined by the nonnegative function

$$\frac{1}{|S|^2} |\widehat{\chi}_S|^2.$$

The question thus arises naturally: does Q tile $\widehat{\mathbb{Z}_n^d}$ in the ordinary sense, i.e. does there exist a set $P \subset \widehat{\mathbb{Z}_n^d}$ such that $Q + P = \widehat{\mathbb{Z}_n^d}$? A famous conjecture of Fuglede [11] concerns exactly this scenario: it states that a set S is spectral if and only if it tiles the group (the conjecture was originally stated in the Euclidean space \mathbb{R}^d but it makes sense in any abelian group).

Several positive partial results were proven concerning special cases of Fuglede's conjecture (see e.g. [16, 17, 19, 27]), before T. Tao [40] showed an example of a spectral set in \mathbb{R}^5 which did not tile the space. Tao's example was based on considerations in the finite group \mathbb{Z}_3^5 . It also implies the existence of such sets in any dimension $d \geq 5$. Subsequently, counterexamples in lower dimensions (4 and 3, respectively) were found by the authors [29, 24]. All these examples are based on considerations in finite groups \mathbb{Z}_n^d and, ultimately, on the existence of certain complex Hadamard matrices.

The other direction of Fuglede's conjecture could not be settled by Tao's arguments. Finally a non-spectral tile in \mathbb{R}^5 was exhibited by the authors [25] by a tricky duality argument, also based on considerations in finite groups. Later, counterexamples in dimensions 4 and 3, respectively, were found in [8, 7]. As of today, Fuglede's conjecture is still open in both directions in dimensions 1 and 2.

As a final remark let us mention another interesting connection of tilings and complex Hadamard matrices. As discussed above, Fuglede's conjecture is not true in general, but it is true in many particular cases. What could this buy for us? We saw from the definition that spectral sets are directly related to complex Hadamard matrices, and the latter are very useful in another branch of mathematics: quantum information theory. Could we use the connection of tilings to spectral sets (which exists in many special cases) to construct new families of complex Hadamard matrices? It turns out that the answer is positive. It turns out [30] that "trivial" tiling constructions unfortunately lead to well-known families of complex Hadamard matrices (so-called Dita-families). However, a non-standard tiling construction of Szabó [37] was used in [30] to produce previously unknown families of complex Hadamard matrices in dimensions 8, 12 and 16.

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