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A Construction Related to the Cosine Problem

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Abstract

We give a constructive proof of the fact that for any sequence of positive integers $n_1, n_2, \ldots, n_N$ there is a subsequence $m_1, \ldots, m_r$ for which

$$-\min_x \sum_1^r \cos m_j x \ge CN,$$

where C is a positive constant. Uchiyama had previously proved the above inequality with the right hand side replaced by $C\sqrt{N}$. We give a polynomial time algorithm for the selection of the subsequence m_j.

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Notation: Let C denote an arbitrary positive constant and $\mathbf{N} = \{1, 2, 3, \ldots\}$ be the set of positive integers. For any real t we write $t^- = \min\{0, t\}$. We denote by $\lfloor t \rfloor$ the greatest integer not greater than t and by $\lceil t \rceil$ the smallest integer not smaller than t.
The Cosine Problem: Chowla [2] has conjectured that for any distinct positive integers $n_1, \ldots, n_N$

$$-\min_x \sum_{j=1}^N \cos n_j x \ge C\sqrt{N}.$$

(There are sequences n_j for which the above minimum is at most $C\sqrt{N}$ in absolute value.) The best result known today in this direction is that of Bourgain [1] who proved that

$$-\min_x \sum_{j=1}^N \cos n_j x \ge C 2^{\log^\epsilon N}$$

for some $\epsilon > 0$.

Uchiyama [4] proved that there is always a subsequence $m_1, \ldots, m_r$ of $n_1, \ldots, n_N$ for which

$$-\min_x \sum_{j=1}^{r} \cos m_j x \geq C\sqrt{N}. \tag{1}$$

He actually proved the stronger statement

$$\frac{1}{2\pi}\int_0^{2\pi} \left| \sum_{j=1}^{r} \cos m_j x \right| dx \geq C\sqrt{N}. \tag{2}$$

In this paper we improve (1).

Theorem 1 *For any sequence $n_1, \ldots, n_N$ of positive integers there is a subsequence $m_1, \ldots, m_r$ such that*

$$-\min_x \sum_{j=1}^{r} \cos m_j x \geq CN. \tag{3}$$

Theorem 1 is an obvious corollary of the more general theorem that follows.

Theorem 2 *Let $w_k \geq 0$ and $w = \sum_1^\infty w_k < \infty$. Then there is a set E of positive integers for which*

$$-\min_x \sum_{k\in E} w_k \cos kx \geq Cw. \tag{4}$$

The essential content of this paper is that the proof of Theorem 2 (and consequently of Theorem 1) we give is constructive. Indeed there is a simple non-constructive proof of our theorem.

Proof of Theorem 2 – Non-constructive: (Odlyzko [3]) Define

$$f(x) = \sum_1^\infty w_k (\cos kx)^-.$$

Then

$$\begin{aligned} \frac{1}{2\pi}\int_0^{2\pi} f(x)dx &= \sum_{k=1}^{\infty} w_k \frac{1}{2\pi}\int_0^{2\pi} (\cos kx)^- dx \qquad (5)\\ &= -\frac{1}{\pi} w. \qquad (6) \end{aligned}$$

Thus there is $x_0 \in [0, 2\pi)$ with $f(x_0) \leq -\frac{1}{\pi} w$. Let $E = \{k \in \mathbf{N} : \cos kx_0 \leq 0\}$. Then obviously

$$\sum_{k\in E} \cos kx_0 \leq -\frac{1}{\pi} w.$$

QED

We now give a constructive proof of Theorem 2 with a worse constant. (See Remark 1 after the proof for the exact meaning of the word "constructive".)

We shall need two lemmas.

Lemma 1 *Let $I_k = (a_k, b_k) \subseteq (0,1)$, $k = 1, 2, \ldots$, be intervals of length at least $\theta > 0$ and w_k be nonnegative weights associated with them. Let also $w = \sum_1^\infty w_k < \infty$. Then there is an interval $J \subseteq (0,1)$, with $|J| = \theta/2$, for which*

$$\sum_{J \subseteq I_k} w_k \geq \frac{1}{2}\theta w. \tag{7}$$

Proof of Lemma 1: Let $m = \lfloor 2/\theta \rfloor$ and $J_\nu = [\nu\theta/2, (\nu+1)\theta/2)$, for $\nu = 0, 1, \ldots, m-1$. Write also $s_\nu = \sum_{a_k \in J_\nu} w_k$. Since $w = \sum_0^{m-2} s_\nu$ there is some $\nu_0 \leq m-2$ for which

$$s_{\nu_0} \geq \frac{w}{m-1} \geq \frac{1}{2}\theta w.$$

Let $J = J_{\nu_0+1}$. Then J satisfies (7) since $a_k \in J_{\nu_0}$ implies $J_{\nu_0+1} \subseteq I_k$. **QED**

The following lemma is a useful special case of Theorem 2.

Lemma 2 *Let $a > 0$, $\sigma > 1$, $\rho \geq 24\sigma$,*

$$E'_j = [\rho^j a, \sigma\rho^j a) \cap \mathbf{N},$$

for $j = 0, 1, 2, \ldots$ and

$$E' = \bigcup_{j=0}^\infty E'_j.$$

Assume also $w_k \geq 0$, $w = \sum_1^\infty w_k < \infty$ and $w_k = 0$ outside E'. Then there is a set $E \subseteq E'$ for which

$$-\min_x \sum_{k \in E} w_k \cos kx \geq \frac{1}{48\sigma} w. \tag{8}$$

Proof of Lemma 2: First observe that in any interval of length at least $2\pi/k$ there is a subinterval of length $2\pi/12k$ in which $\cos kx \leq -1/2$. According to this observation, for all $k \in E'_0$ there is an interval I_k contained in $(0, 2\pi/a)$, of length at least $2\pi/12\sigma a$, in which $\cos kx \leq -1/2$. By Lemma 1 ($\theta = 1/12\sigma$) there is an interval $J_0 \subseteq (0, 2\pi/a)$ of length $2\pi/24\sigma a$ for which

$$\sum_{J_0 \subseteq I_k} w_k \geq \frac{1}{24\sigma} \sum_{k \in E'_0} w_k. \tag{9}$$

Let $E_0 = \{k \in E'_0 : J_0 \subseteq I_k\}$. Then

$$\sum_{k \in E_0} w_k \cos kx \leq -\frac{1}{48\sigma} \sum_{k \in E'_0} w_k, \text{ for all } x \in J_0. \tag{10}$$

Similarly we can find an interval $J_1 \subseteq J_0$, with $|J_1| = 2\pi/12\sigma\rho a$, and $E_1 \subseteq E'_1$, such that

$$\sum_{k \in E_1} w_k \cos kx \leq -\frac{1}{48\sigma} \sum_{k \in E'_1} w_k, \text{ for all } x \in J_1.$$

This is possible since $\rho \geq 24\sigma$ and therefore J_0 is big enough to accomodate all frequencies in E'_1. In the same fashion we define $J_2 \supseteq J_3 \supseteq \ldots$, and $E_2, E_3, \ldots$. Finally we set $E = \bigcup_0^\infty E_j$. It follows that (8) is true. **QED**

We can now complete the proof of the theorem.

Proof of Theorem 2 – Constructive: Let $\sigma = 2$, $\rho = 64$ and write for $\nu = 0, \ldots, 5$

$$A_\nu = \bigcup_{j=0}^{\infty} [\rho^j \sigma^\nu, \rho^j \sigma^{\nu+1}) \cap \mathbf{N}.$$

Since $\mathbf{N} = \bigcup_0^5 A_\nu$ there is some ν_0 for which

$$\sum_{k \in A_{\nu_0}} w_k \geq \frac{1}{6} w. \tag{11}$$

An application of Lemma 2 with $\sigma = 2$, $\rho = 64$, $a = 1$ and the collection of w_k for $k \in A_{\nu_0}$ furnishes a set $E \subseteq A_{\nu_0}$ for which

$$-\min_x \sum_{k \in E} w_k \cos kx \geq \frac{1}{6 \cdot 48 \cdot 2} w.$$

QED

Remarks

1. The simple proof of Theorem 1 mentioned can of course be made constructive by looking for an x that satisfies

$$\sum_1^N (\cos n_k x)^- \leq -\frac{1}{2\pi} N$$

among the points $x_j = jh$, for $j = 0, \ldots, \lfloor 1/h \rfloor$. But h has to be smaller than $C n_N^{-1}$ and this leads to an algorithm which in the worst case takes time exponential in the size of the input (which is considered to be the number of binary digits required to write down all $n_1, \ldots, n_N$). For example if $n_N = 2^N$ then the algorithm needs time at least $C 2^N$ but the size of the input is at most N^2.

In contrast, our construction takes time which is polynomial in the size of the input (in other words, polynomial in $N \log n_N$). Assume that we are given N positive integers $n_1 \leq \cdots \leq n_N$ and let $L = \lceil \log_2 n_N \rceil$. Define $w_j = |\{k \in \mathbf{N} : j = n_k\}|$. The algorithm we described consists of the following steps. The notation of Lemma 2 is used throughout.

1. Find for which $\nu_0 \in \{0, \ldots, 5\}$ inequality (11) is true.
2. Construct the sequence of intervals $J_0 \supseteq J_1 \supseteq \cdots$ and the sequence of sets $E_0, E_1, \ldots$. This proceeds inductively. Having constructed the interval J_{m-1} and the set E_{m-1} we
   a. construct the intervals I_{n_k} for all $n_k \in E'_m$,
   b. find (as described in Lemma 1) a subinterval J_m of J_{m-1} which is big and is contained in many of the I_{n_k}'s. The set E_m consists of those $n_k \in E'_m$ for which $J_m \subseteq I_{n_k}$.

   Notice that the sequences J_m and E_m have length $O(L)$.

After observing that we never need to perform arithmetic with more than $O(L)$ binary digits, it is easy to see that all the above can be carried out in time $O(N \cdot L^2)$, since an algebraic operation on two numbers, with $O(L)$ binary digits each, takes $O(L^2)$ time.

2. Uchiyama's proof of (1) is probabilistic. We give an even simpler constructive proof of (1). (Of course Uchiyama proved the stronger statement (2) about the L^1 norm of a subseries.) Assume $n_1 \le n_2 \le \cdots \le n_N$ and let ρ be any fixed number between 2 and 3, say $\rho = 5/2$. Observe that if $n_N \le \rho n_1$ then

$$-\min_x \sum_{j=1}^{N} \cos n_j x \ge CN,$$

as can be seen by evaluating the function $\sum_{j=1}^{N} \cos n_j x$ for $x = (\pi/2 + \epsilon)/n_1$, where $\epsilon = \epsilon(\rho)$ is a small positive constant.

Let $\lambda_1 = n_1$ and define $\lambda_k \in \{n_1, \ldots, n_N\}$ recursively by

$$\lambda_k = \min \{n_j : n_j > \rho\lambda_{k-1}\} \cup \{n_N\}.$$

Let L be the length of the sequence λ_k. That is let λ_L be the first λ equal to n_N. Then either $L \ge \sqrt{N}$ or there is some k for which the set

$$A = \{n_j : \lambda_k \le n_j < \lambda_{k+1}\}$$

has more than $\sqrt{N}$ elements. In the first case we have

$$-\min_x \sum_{j=1}^{L-1} \cos \lambda_j x \ge CL \ge C\sqrt{N},$$

since the λ_j's form a lacunary sequence with ratio $\rho > 2$. Otherwise, according to the above observation, we have

$$-\min_x \sum_{n_j \in A} \cos n_j x \ge C|A| \ge C\sqrt{N},$$

which completes the proof.

3. It is easy to see that Theorem 2 holds also for complex w_k, with $w = \sum |w_k| < \infty$ and writing e^{ikx} in place of $\cos kx$. Also the minimum in (4) has to be interpreted as the minimum (or maximum) of the real part.

References

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